Notions of size or cardinality

We say sets $X, Y$ have the same size (cardinality) if there exist a bijection between them, and write $|X| = |Y|$. One-to-one correspondence $\equiv$ injective & surjective

If $f : X \rightarrow Y$, then

(i) $f$ is surjective means

\[ (\forall x \in X) \; \; \; \; \exists y \in Y \; \; \; \; f(x) = y \]

(ii) $f$ is injective means

\[ (\forall x, y \in X) \; \; \; \; f(x) = f(y) \Rightarrow x = y \]

There is a natural duality between these notions that is not clear from the definitions but becomes transparent in category theory.
e.g. If \( x = \{\text{students milling around outside}\} \)

\( y = \{\text{seats in lecture theatre}\} \)

\( f : x \rightarrow y, \ x \mapsto (\text{seat occupied by } x \text{ in lecture}) \)

Then \( f \) is one-one if different students occupy different seats, and onto if every seat is occupied in the lecture.

Put \( [n] = \{1, 2, \ldots, n\} \) for \( n \in \mathbb{Z}^+ \)

and \( [0] = \emptyset \) (empty set).

Call a set \( x \) finite if

\[(\exists n \in \mathbb{Z}^+ \cup \{0\}) \quad 1 \times 1 = |[n]| \]

in which case we also say \( 1 \times 1 \rightarrow n \)

\( x \) has size \( n \). \[\underline{x \text{ has size } n} \]

Call \( x \) infinite if \( x \) is not finite.
Every finite X is infinite if

\[ \exists \text{ injective mapping } \mathbb{Z}^+ \rightarrow X \]

Proof: exercise by induction.

If there is an injective function \( A \rightarrow B \), then we write \( |A| \leq |B| \).

Theorem (Schröder-Bernstein):
If \( A \) and \( B \) are sets and
\[ |A| \leq |B| \leq |A| \]
then \( |A| = |B| \).

Proof: very difficult exercise.

Problem is handling infinite sets.
If \( f : A \rightarrow B \) is injective and \( g : B \rightarrow A \)

is also injective, then we have a faithful
"snapshot" of \( A \) inside \( B \) and \( B \) inside \( A \),
but it is not obvious how to glue things
together to get a bijection \( A \rightarrow B \)

Hint: alternate "taking snapshots" back and forth forever and try to control bits & pieces
that are missing.
Consider the real line $\mathbb{R}$.

Then $2^\mathbb{N} = \{0, 1, 2, \ldots\} \subset \mathbb{R}$, and the map $x \mapsto x$ is injective $2^\mathbb{N} \rightarrow \mathbb{R}$, so $|2^\mathbb{N}| \leq |\mathbb{R}|$.

**Cell a set $X$ countable if $X$ is finite or $|X| = |\mathbb{N}|$.**

Put $N = \{0, 1, 2, \ldots\} = 2^\mathbb{N} \cup \{0, 1, 2, \ldots\}$.

Then $N$ is countable because $x \mapsto x + 1$ is a bijection $N \rightarrow 2^\mathbb{N}$, so that $|N| = |2^\mathbb{N}|$.

**Exercise:** Find a bijection between $\mathbb{Z}$ and $2^\mathbb{N}$, so $|\mathbb{Z}| = |2^\mathbb{N}|$. 


Theorem (Cont'd): \( \mathbb{R} \) is uncountable.

Proof: Suppose \( \mathbb{R} \) is countable, so there exists a bijection \( f: \mathbb{Z}^+ \to \mathbb{R} \).

Thus we may "list" the real numbers:

\[
f(1) = \cdots d_1 \, d_{11} \, d_{12} \, d_{13} \, d_{14} \cdots
\]

\[
f(2) = \cdots d_2 \, d_{21} \, d_{22} \, d_{23} \, d_{24} \cdots
\]

\[
f(3) = \cdots d_3 \, d_{31} \, d_{32} \, d_{33} \, d_{34} \cdots
\]

\[
f(4) = \cdots d_4 \, d_{41} \, d_{42} \, d_{43} \, d_{44} \cdots
\]

\[
f(n) = \cdots d_n \, d_{n1} \, d_{n2} \, d_{n3} \, d_{n4} \cdots
\]

where \( (n \in \mathbb{Z}^+) \) \( e_i \neq d_{ij} \) and \( e_i \in \{0, 9\} \).
Then \( x \in \mathbb{R} \) so \( x = f(c) \) for some \( c \in \mathbb{Z}^+ \).

Suppose \( 0 < a < e_r \)...

... and differ at some place

so \( x \neq f(n) \), a contradiction.

Hence no such bijection exists and \( \mathbb{R} \)

is uncountable.

Note: Decimal expansions of real numbers are unique except for
repeated 0s and 9s.

e.g. \( 0.999... = 1.000... \)

We avoid this issue by choosing \( \epsilon \neq 0, 9 \)