Propositional Calculus continued

Recall the truth table for implication:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P → Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>T</td>
<td>F</td>
<td>F</td>
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<tr>
<td>F</td>
<td>T</td>
<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

and we may regard \( P → Q \) as an abbreviation for \( \neg P \lor Q \), because they are logically equivalent (having some truth values):

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>\neg P</th>
<th>\neg P \lor Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
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<td>F</td>
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<td>T</td>
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</tbody>
</table>

The truth values in \( \bigcirc \) were "tossed" on us by natural interpretations of:

- converse: \( Q \rightarrow P \neq P \rightarrow Q \)
- contrapositive: \( \neg Q \rightarrow \neg P \equiv P \rightarrow Q \)
Implication gives rise to the most important rule of deduction called modus ponens:

\[
\begin{align*}
P & \rightarrow \neg Q \\
\therefore \ P & \rightarrow \neg Q
\end{align*}
\]

The contrapositive gives rise to another important rule of deduction called modus tollens:

\[
\begin{align*}
\neg Q & \rightarrow \neg P \\
\therefore \ P & \rightarrow \neg Q
\end{align*}
\]

Note: it is fallacious to argue

\[
\begin{align*}
\neg P & \rightarrow \neg Q \\
\therefore \ P & \rightarrow \neg Q
\end{align*}
\]

(Why?)
Double implication (if and only if):

\[ P \iff Q \iff (P \implies Q) \land (Q \implies P) \]

To see the following truth table:

<table>
<thead>
<tr>
<th>P &amp; Q</th>
<th>P = Q</th>
<th>Q \implies P</th>
<th>(P = Q) \land (Q \implies P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T &amp; T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T &amp; F</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F &amp; T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F &amp; F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

ie. \( P \iff Q \) is true precisely when \( P \) and \( Q \) have identical truth values.
A theorem in propositional calculus is a statement that always has T as truth value, regardless of the truth values of the variables.

—and so produces a column of T's in its truth table.

—to test if something is a theorem, assume it is false and work backwards looking for a contradiction or a counterexample.
Examples:

1) \[(P \implies Q) \land (Q \implies R) \implies (P \implies R)\]

\[
\begin{array}{cccccc}
0 & 1 & 1 & 0 & 0 & 0 \\
\end{array}
\]

10, 11 yield a contradiction so we have a theorem.

2) \[(P \lor Q) \implies R \implies [(P \implies R) \lor (Q \implies R)]\]

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

2, 11, 12 yield a contradiction so we have a theorem.

3) \[\[(P \implies R) \lor (Q \implies R) \implies [(P \lor Q) \implies R]\]

\[
\begin{array}{cccccc}
0 & 1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

9, 10, 6, 2, 11, 7, 1, 8, 4, 1, 3, 5

\(P = T, Q = F, R = F\) produce a counterexample so we do not have a theorem.

Check:

\[\[(P \implies R) \lor (Q \implies R) \implies [(P \lor Q) \implies R]\]

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
\[(P \Rightarrow R) \land (Q \Rightarrow R) \Rightarrow (P \lor Q) \Rightarrow R\]

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>(P \lor Q)</th>
<th>((P \land Q) \Rightarrow R)</th>
</tr>
</thead>
<tbody>
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<td>T</td>
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</tbody>
</table>

2, 10 yield a contradiction, so \(x \) is wrong

6, 12 yield a contradiction, so we have a theorem

Earlier in this case:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>(P \lor Q)</th>
<th>((P \land Q) \Rightarrow R)</th>
</tr>
</thead>
<tbody>
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4, 12, 13 yield a contradiction, so again confirming that we have a theorem.
**Well-formed formulae (wffs)**

**Inductive definition:**

(i) Propositional variables

\[ P_1, P_2, R_1, \ldots, P_i, P_{i+1}, P_3, P_4, \ldots \]

are wffs.

(ii) If \( x \) and \( y \) are wffs then so are

- (a) \( \neg x \)
- (b) \( x \land y \)
- (c) \( x \lor y \)
- (d) \( x \rightarrow y \)
- (e) \( x \leftrightarrow y \)

(iii) A string of symbols is not a wff unless formed by a finite number of applications of (i) and (ii).
In practice, find outer brackets are "invisible" and many logicians use precedence of operators to render inner brackets to avoid congestion.

Examples:

(1) \( P, Q, R \) are wffs, so

\[ P \lor Q, P \Rightarrow Q, Q \Rightarrow R \quad \text{are wffs}, \]

so \( (P \Rightarrow Q) \land (Q \Rightarrow R) \) and \( (P \lor Q) \Rightarrow R \) are wffs,

so \[ (P \Rightarrow Q) \land (Q \Rightarrow R) \Rightarrow [(P \lor Q) \Rightarrow R] \quad \text{is a wff}. \]

(2) \[ (P \Rightarrow Q) \lor (\neg Q) \]

can be seen to be a wff by "peeling back layers; built from \( (P \Rightarrow Q) \lor (\neg Q) \) and \( (\neg (\neg P)) \land Q \)

\[ \vdash \quad P \Rightarrow Q, Q \Rightarrow (\neg P), \neg Q \]

\[ \vdash \quad P, Q, Q \Rightarrow (\neg P), \neg Q \]

finally all built from \( P \lor Q \).
Exercise: Prove the following by induction.

(i) For formal wffs where no brackets have been omitted:

1. The number of left brackets equals the number of right brackets in any wff.
2. There are no wffs using exactly 2, 3 or 6 symbols, but any other number of symbols is possible.
3. Let $W$ be a wff and let $b$ be the number of times a binary connective ($\land, \lor, \rightarrow, \leftrightarrow$) occurs. Let $p$ be the number of times propositional variables occur. Then $p = b + 1$. 