A Metatheorem about Propositional Calculus

The Propositional Calculus has 10 rules of derivation:

\[ \text{A, MP, MT, DN, CP, } \land I, \land E, \lor I, \lor E, \text{ RAA} \]

and some extra bells and whistles:

\[ \text{Det, TI, SI, TI(S), SI(S)} \]

(Note TI is a special case of SI.)

There is another sophisticated derived rule that enables huge flexibility and possible abbreviations.

Meta-theorem: Let \( V \) be a subwff of a wff \( W \) and suppose \( V' \) is another wff such that \( V \rightarrow V' \).

Let \( W' \) be the wff obtained by replacing \( V \) by \( V' \) in the construction of \( W \).

Then \( W \rightarrow W' \).
e.g. Since \( \sim \sim X \vdash X \)

for all wfts \( X \), we may "cancel" consecutive pairs of double tildes wherever we see them (or introduce them) without changing the "provability" of any given wft.

**E.g.** If \( \sim X \forall Y \) appears as a subwft

we may replace it by \( X \Rightarrow Y \) since

\[
X \Rightarrow Y \vdash \sim X \forall Y
\]

**E.g.** If \( \forall X \forall Y \) appears as a subwft

we may replace it by \( \forall X \Rightarrow \forall Y \) or \( \forall Y \Rightarrow X \)

since

\[
\forall X \forall Y \vdash \sim \forall X \forall Y \Rightarrow \forall X \sim \forall Y \Rightarrow X
\]

(Note \( \Rightarrow \) is transitive.)
Recall $W$ is a \underline{wff} (well-formed formula) if
\begin{enumerate}
\item $W$ is a propositional variable,
\item $W = \neg x, (x \land y), (x \lor y), (x = y), (x \equiv y)$
\end{enumerate}
for some wff $x, y$ of shorter length.

A \underline{subwff} of $W$ is any wff that appears in the construction of $W$ at any stage.

\textit{e.g.}, let $W$ be the wff
\[ \neg ( (P \lor \neg Q) \land (\neg (P \land (\neg Q \Rightarrow \neg P)))) \],
then the subwffs are precisely $W$ and
\[ (P \lor \neg Q) \land (\neg (P \land (\neg Q \Rightarrow \neg P))) \],
\[ P \lor \neg Q, \neg (P \land (\neg Q \Rightarrow \neg P)) \],
\[ P, \neg Q, P \land (\neg Q \Rightarrow \neg P), Q, P, \neg Q \Rightarrow \neg P, \neg Q, \neg P, Q, P. \]

Now $\boxed{P \land (\neg Q \Rightarrow \neg P) \vdash P \lor Q}$

(exercise)

So $\boxed{W \vdash \neg (P \lor Q) \land (\neg (P \land \neg Q))}$

by the Meta- theorem using \textit{resolution}. 
But \[ \sim (P \land Q) \quad \Rightarrow \quad \sim P \lor \sim Q \]
(exercise)

So \[ \sim P \lor \sim Q \quad \Rightarrow \quad \sim (P \land Q) \]
by the Metatheorem

But \[ (P \lor Q) \land (\sim P \land Q) \quad \Rightarrow \quad \sim Q \]
(exercise)

So \[ \sim Q \quad \Rightarrow \quad \sim P \lor \sim Q \]
by the Metatheorem

Proof of the Metatheorem:
If \( V = W \) then \( V' = W' \) and it is immediate that \( W \models W' \). If \( W \) is a propositional variable then \( V \models W \) and we are done by the previous observation, which states our induction (on the length of \( W \)).

Suppose \( W \) is not a variable, so either

(i) \( W = \sim X \) or
(ii) \( W = X + Y \)

where \( X, Y \) are wffs of shorter length and
\[ \sim (\lor) \Rightarrow \lor (\equiv) \]
We may suppose $V \neq W$.

Case (1): $W = \sim X$ and $V$ is a subwff of $X$.

By an inductive hypothesis,

$\boxed{X \vdash X'}$,

where $X'$ is the result of replacing $V$ by $V'$ in $X$. The following is a proof of $W \vdash W' = \sim X'$:

1. (1) $\sim X$ A
2. (2) $X'$ A
2. (3) $X$ 2 SI ⊢
1, 2 (4) $X \land \sim X$ 1, 3 ∧I
1 (5) $\sim X'$ 2, 4 RAA

A similar proof yields $W' \vdash W$ and so $W \vdash W'$ as required.
Case (ii): \( W = x \cdot y \) and \( V \) is a subwff of \( x \) or \( y \).

Without loss of generality.

Subcase (a): \( W = x \cdot y \) and WLOC \( V \) is a subwff of \( x \), and \( \otimes \) holds as before.

The following is a proof of \( W \vdash W' = x' \cdot y' \):

1. \( (1) \; x \cdot y \; A \)
2. \( (2) \; x \; 1 \cdot E \)
3. \( (3) \; x' \; 2 \; SI \otimes \)
4. \( (4) \; y \; 1 \cdot E \)
5. \( (5) \; x' \cdot y' \; 3,4 \cdot I \)

and similarly \( W' \vdash W \) to \( W \vdash W' \).

Subcase (b): \( W = x \cdot y \) and WLOC \( V \) is a subwff of \( x \), and \( \otimes \) holds as before.

The following is a proof of \( W \vdash W' = x' \cdot y' \):

1. \( (1) \; x' \cdot y' \; A \)

   "case (i)"

   2. \( (2) \; x' \; A \)
3. \( (3) \; x' \cdot y' \; 2 \; SI \otimes \)
4. \( (4) \; x' \cdot y' \; 3 \cdot I \)

   "case (ii)"

   5. \( (5) \; y \; A \)
6. \( (6) \; x' \cdot y' \; 5 \cdot E \)

and similarly \( W' \vdash W \) to \( W \vdash W' \).
Subcase (c): $w = x \Rightarrow y$.

Suppose first $w$ is a subwell of $x$ and

0 holds as before.

The following is a proof of $w \vdash w' = x \Rightarrow y$.

1. (1) $x \Rightarrow y$ A
2. (2) $x'$ A
3. $x \Rightarrow y$, 2, SI ⊢
4. $y \in x$, 1, 3, MP
5. $x' \Rightarrow y$, 2, 4, CP

and similarly $w \vdash w'$ so $w \vdash w'$ √

Now suppose $w$ is a subwell of $y$ and $y'$ is the result of replacing $w$ by $x'$.

In the inductive hypothesis, yields

$y \vdash y'$

The following is a proof of $w \vdash w' = x \Rightarrow y'$.

1. (1) $x \Rightarrow y$ A
2. (2) $x$ A
3. $x \Rightarrow y$, 2, MP
4. $y \Rightarrow y'$, 3, SI √
5. $x \Rightarrow y'$, 2, 4, CP
and similarly $W' - W$ so $\boxed{W' + W'}$.

Subcase (d): $W = x \ominus y$ and $w \not\in \om$. $V$ is a subwff of $x$ and $\neg \beta$ holds as before.

The following is a proof of $W - W' = x \ominus y$.

\begin{align*}
1 & \quad (1) \quad x \ominus y \quad A \\
2 & \quad (2) \quad (x \ominus y) \land (y = x) \\
3 & \quad (3) \quad x = y \\
4 & \quad (4) \quad y = x \\
5 & \quad (5) \quad x' \equiv y \\
6 & \quad (6) \quad y \equiv x' \\
7 & \quad (7) \quad (x' \equiv y) \land (y = 1 \cdot x') \\
8 & \quad (8) \quad x' \ominus y \\
\end{align*}

and similarly $W' - W$ so $\boxed{W' + W'}$.

This completes the proof of the metatheorem by induction (on the length of $W$).