Remarks about sizes of large sets

Recall two sets $X$ and $Y$ have the same size (cardinality) if there exists a bijection $f : X \to Y$ (so $f^{-1} : Y \to X$), and we write

$$|X| = |Y|.$$

For example:

$f : \mathbb{N} \to 2^+ \quad \text{by} \quad x \mapsto x + 1 \quad (\forall x \in \mathbb{N})$

is a bijection, so $|\mathbb{N}| = |2^+|.$

Recall $|\emptyset| = 0$ and $|\mathbb{N}| = 1, 2, \ldots, n^3$ for $n \in 2^+.$

Call a set $X$ infinite if there is no bijection between $X$ and $\{n\}$ for any $n \in \mathbb{N} = \{0, 1, \ldots\}.$

(Think of $\{n\}$ as a prototype for a finite set.)

Claim: If $X$ is infinite then there exists an injective map $f : 2^+ \to X.$
Post: Let $x$ be infinite. We show by induction.

There exist distinct $f(1), f(2), \ldots, f(n) \in x$ for each $n \in 2^+$.

Since $x$ is infinite, $x \not\subseteq [0]^\omega$, so we may choose $f(1) \in x$, establishing $\forall n \in 1$, starting an induction.

Suppose (as inductive hypothesis) that $\forall n$ holds for given $n \in 2^+$. (We show $\forall n$ holds with $n$ replaced by $n+1$.)

If $x = \{ f(1), f(2), \ldots, f(n) \}$, then $f: \{ n \} \to x$, $i \mapsto f(i)$ for $i = 1, \ldots, n$ is a bijection, contradicting that $x$ is infinite.

Hence there exists some $f(n+1) \not\in x \setminus \{ f(1), \ldots, f(n) \}$, so that $f(1), f(2), \ldots, f(n), f(n+1)$ are distinct, establishing the inductive step. Thus $\forall n$ holds for all $n \in 2^+$. This yields an injective map $f: 2^+ \to x$.\qed
Thus, inside any infinite set \( X \), no matter how

bizarre, we can find a "copy" of \( 2^+ \)

\[
\{ f(1), f(2), f(3), \ldots, f(n), \ldots \}
\]

We can exploit this to make room for other

things, e.g.,  

- ...

**Corollary.** If \( X \) is infinite and \( Y \) is finite

then there is a bijection \( g: X \cup Y \mapsto X \).

**Proof.** Suppose \( X \) is infinite, \( Y \) is finite

and \( \emptyset \subseteq X \cap Y = \emptyset \) (\( X \) and \( Y \) disjoint)

If \( Y = \emptyset \)

then \( X \cup Y = X \), so \( g = \text{id} : X \cup Y \mapsto X \)

serves as a bijection.

Suppose \( Y \neq \emptyset \), so \( Y = \{y_1, \ldots, y_n\} \) for some \( n \in \mathbb{N} \).

all distinct elements.
From the earlier claim,

there exists an injective \( f : \mathbb{R}^+ \to X \).

Define \( g : X \cup Y \to X \) by

\[
g(z) = \begin{cases} 
  z & \text{if } z \in X \setminus \text{im} f \\
  f(x+i) & \text{if } z = f(x) + \text{im} f \\
  f(i) & \text{if } z = y_i \\
\end{cases}
\]

for \( x, y \in X \).

Then \( g \) is clearly one-to-one & onto, i.e. bijective.

**Corollary:** There exists an injection \( h : [0,1] \to (0,1) \).

**Proof:** \([0,1] = (0,1) \cup [0,1]^c\).

(We are not claiming \( h \) preserves any structure.
For example, \( h \) cannot be a topological isomorphism.
Since \([0,1] \) is compact whilst \((0,1) \) is not.)
**Corollary:** $\mathcal{P}(\mathbb{Z}^+) = |\mathbb{R}|$

**Proof:**

Let $k : \mathbb{R} \to (0,1)$ where

$$k(x) = \frac{1}{1 + e^{-x}}$$

so $k$ is one-one & onto, a bijection.

From the previous corollary, $h : [0,1] \to (0,1)$ is a bijection, so

$$h^{-1} : (0,1) \to [0,1]$$

is also a bijection. From the First Assignment, a bijection $\ell : \mathcal{P}(\mathbb{Z}^+) \to [0,1]$, so define a bijection $\ell \circ h^{-1} k : \mathbb{R} \to \mathcal{P}(\mathbb{Z}^+)$, which proves $|\mathcal{P}(\mathbb{Z}^+)| = |\mathbb{R}|$. □
Another important set-theoretic construction:

The Cartesian product of sets $A$ and $B$ is

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

(so-called because Descartes introduced the idea when $A = B = \mathbb{R}$).

Exercises: (1) $A \times \emptyset = \emptyset \times B = \emptyset$

(2) If $|A| = m$, $|B| = n$ then

$$|A \times B| = mn$$

for $m, n \in \mathbb{Z}$.

(3) If $|A| = \mathbb{Z}^+$ (countably infinite)

and $B$ is finite then $|A \times B| = |\mathbb{Z}^+|$.

(4) $\mathbb{Z}^+ \times \mathbb{Z}^+ = |\mathbb{Z}^+|$

(5) If $A$ and $B$ are countable then

$A \times B$ is countable.

*** (6) If $A$ and $B$ are infinite and $|A| \leq |B|$

then $|A \times B| = |B|$. Probably beyond the scope of this course.