1. Denote the number of symbols in a positive wff \( W \) by \( \#W \). Let \( L \) denote the set of lengths of all positive wffs and

\[
K = \{1 + 4k \mid k \in \mathbb{N}\}.
\]

We claim that \( K = L \). To see that \( K \subseteq L \), define positive wffs \( W_n \) inductively by

\[
W_0 = P \quad \text{and} \quad W_{n+1} = (W_n \lor P) \quad \text{for } n \geq 0,
\]

where \( P \) is a fixed propositional variable. We verify that \( \#W_k = 1 + 4k \) for each \( k \in \mathbb{N} \) by induction. Certainly, \( \#W_0 = 1 \), which starts an induction. If we assume, as inductive hypothesis, that \( \#W_n = 1 + 4n \), for \( n \geq 0 \), then

\[
\#W_{n+1} = \#W_n + 4 = 1 + 4n + 4 = 1 + 4(n+1),
\]

completing the inductive step. Thus \( \#W_k = 1 + 4k \) for each \( k \in \mathbb{N} \), proving \( K \subseteq L \).

To see that \( L \subseteq K \), we prove, by induction on the length of a positive wff \( W \), that \( \#W \in K \). If \( \#W = 1 \) then certainly \( 1 \in K \), which starts an induction. Suppose that \( \#W > 1 \), so

\[
W = (U \lor V) \quad \text{or} \quad W = (U \land V)
\]

for some positive wffs \( U, V \) of smaller length. By an inductive hypothesis, \( \#U, \#V \in K \), so that

\[
\#U = 1 + 4k \quad \text{and} \quad \#V = 1 + 4\ell
\]

for some \( k, \ell \in \mathbb{N} \). But then

\[
\#W = \#U + \#V + 3 = 1 + 4k + 1 + 4\ell + 3 = 1 + 4(k + \ell + 1) \in K,
\]

completing the inductive step. Hence \( L \subseteq K \), and so \( K = L \).

(8 marks)

2. Let \( W = W(X_1, \ldots, X_n) \) be a positive wff built using propositional variables \( X_1, \ldots, X_n \). If \( \#W = 1 \) then \( n = 1 \) and \( W = X_1 \), so that if \( V(X_1) = T \) then certainly \( V(W) = T \), which starts an induction. Suppose that \( \#W > 1 \) so that \( W = (R * S) \), where * is one of \( \land, \lor, \Rightarrow \) or \( \Leftrightarrow \), and

\[
R = R(X_1, \ldots, X_n) \quad \text{and} \quad S = S(X_1, \ldots, X_n)
\]

(8 marks)
are positive wffs of smaller length (regardless of whether all propositional variables $X_1, \ldots, X_n$ actually appear in each of $R$ and $S$). Suppose that

$$V(X_1) = \ldots = V(X_n) = T.$$ 

By an inductive hypothesis applied to each of $R$ and $S$, we have

$$V(R) = T \quad \text{and} \quad V(S) = T.$$ 

By the first row of the truth tables for each of $\land$, $\lor$, $\Rightarrow$ and $\iff$, we have $V(W) = T$. The result now follows by induction. 

(6 marks)

3. Observe that, by de Morgan’s Laws and other logical equivalences,

$$W = \sim \left[ (P \Rightarrow \sim Q) \land (R \land (P \lor Q)) \right]$$

$$\equiv \sim (P \Rightarrow \sim Q) \lor \sim (R \land (P \lor Q))$$

$$\equiv (P \land \sim Q) \lor [\sim R \lor \sim (P \lor Q)]$$

$$\equiv (P \land \sim Q) \lor [\sim R \lor (\sim P \land \sim Q)]$$

$$\equiv \sim R \lor [(P \land Q) \lor (\sim P \land \sim Q)]$$

$$\equiv R \Rightarrow [(P \land Q) \lor (\sim P \land \sim Q)]$$

$$\equiv R \Rightarrow (P \iff Q),$$

which is a positive wff. 

(6 marks)

4. (a) Claim: $(\forall x)((F(x) \lor G(x)) \Rightarrow H(x)), \ (\exists x) \sim H(x) \vdash (\exists x) \sim F(x)$

Proof:

<p>| | | | | | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>(1)</td>
<td>$(\forall x)((F(x) \lor G(x)) \Rightarrow H(x))$</td>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(2)</td>
<td>$(\exists x) \sim H(x)$</td>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(3)</td>
<td>$\sim H(a)$</td>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(4)</td>
<td>$(F(a) \lor G(a)) \Rightarrow H(a)$</td>
<td>1 $\forall$ E</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1, 3</td>
<td>(5)</td>
<td>$\sim (F(a) \lor G(a))$</td>
<td>3, 4 MT</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(6)</td>
<td>$F(a)$</td>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(7)</td>
<td>$F(a) \lor G(a)$</td>
<td>6 $\lor$ I</td>
<td></td>
<td></td>
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<tr>
<td>1, 3, 6</td>
<td>(8)</td>
<td>$(F(a) \land G(a)) \lor (F(a) \lor G(a))$</td>
<td>5, 7 $\land$ I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1, 3</td>
<td>(9)</td>
<td>$\sim F(a)$</td>
<td>6, 8 RAA</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1, 3</td>
<td>(10)</td>
<td>$(\exists x) \sim F(x)$</td>
<td>9 $\exists$ I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1, 2</td>
<td>(11)</td>
<td>$(\exists x) \sim F(x)$</td>
<td>2, 3, 10 $\exists$ E</td>
<td></td>
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</tr>
</tbody>
</table>
(b) Claim: \((\forall x)(F(x) \Rightarrow G(x)) \vdash ((\exists x) \sim G(x)) \Rightarrow ((\exists x) \sim F(x))\)

Proof:

1. \((1)\) \((\forall x)(F(x) \Rightarrow G(x))\) \hspace{1cm} A
2. \((2)\) \((\exists x) \sim G(x)\) \hspace{1cm} A
3. \((3)\) \(\sim G(a)\) \hspace{1cm} A
4. \((4)\) \(F(a) \Rightarrow G(a)\) \hspace{1cm} 1 \ \forall \ E
5. \((5)\) \(\sim F(a)\) \hspace{1cm} 3, 4 \ MT
6. \((6)\) \((\exists x) \sim F(x)\) \hspace{1cm} 5 \ \exists \ I
7. \((7)\) \((\exists x) \sim F(x)\) \hspace{1cm} 2, 3, 6 \ \exists \ E
8. \((8)\) \(((\exists x) \sim G(x)) \Rightarrow ((\exists x) \sim F(x))\) \hspace{1cm} 2, 7 \ CP

(c) Claim: \((\forall x)(F(x) \Rightarrow \sim G(x)) \vdash \sim ((\exists x)(F(x) \land G(x)))\)

Proof:

1. \((1)\) \((\forall x)(F(x) \Rightarrow \sim G(x))\) \hspace{1cm} A
2. \((2)\) \((\exists x)(F(x) \land G(x))\) \hspace{1cm} A
3. \((3)\) \(F(a) \land G(a)\) \hspace{1cm} A
4. \((4)\) \(F(a) \Rightarrow \sim G(a)\) \hspace{1cm} 1 \ \forall \ E
5. \((5)\) \(F(a)\) \hspace{1cm} 3 \ \land \ E
6. \((6)\) \(G(a)\) \hspace{1cm} 3 \ \land \ E
7. \((7)\) \(\sim G(a)\) \hspace{1cm} 4, 5 \ MP
8. \((8)\) \(G(a) \land \sim G(a)\) \hspace{1cm} 6, 7 \ \land \ I
9. \((9)\) \(\sim (\forall x)(F(x) \Rightarrow \sim G(x))\) \hspace{1cm} 1, 8 \ RAA
10. \((10)\) \(\sim (\forall x)(F(x) \Rightarrow \sim G(x))\) \hspace{1cm} 2, 3, 9 \ \exists \ E
11. \((11)\) \((\forall x)(F(x) \Rightarrow \sim G(x))\) \hspace{1cm} 1, 10 \ \land \ I
12. \((12)\) \(((\exists x)(F(x) \land G(x))\) \hspace{1cm} 2, 11 \ RAA

(d) Claim: \((\exists x)\left(F(x) \land (\forall y)(G(y) \Rightarrow K(x, y))\right)\), \((\forall x)(F(x) \Rightarrow (\forall y)(H(y) \Rightarrow \sim K(x, y)))\) \(\vdash (\forall x)(G(x) \Rightarrow \sim H(x))\)

Proof:

1. \((1)\) \((\exists x)\left(F(x) \land (\forall y)(G(y) \Rightarrow K(x, y))\right)\) \hspace{1cm} A
2. \((2)\) \((\forall x)\left(F(x) \Rightarrow (\forall y)(H(y) \Rightarrow \sim K(x, y))\right)\) \hspace{1cm} A
3. \((3)\) \(G(b)\) \hspace{1cm} A
4. \((4)\) \(F(a) \land (\forall y)(G(y) \Rightarrow K(a, y))\) \hspace{1cm} A
5. \((5)\) \(F(a) \Rightarrow (\forall y)(H(y) \Rightarrow \sim K(a, y))\) \hspace{1cm} 2 \ \forall \ E
6. \((6)\) \(F(a)\) \hspace{1cm} 4 \ \land \ E
7. \((7)\) \((\forall y)(G(y) \Rightarrow K(a, y))\) \hspace{1cm} 4 \ \land \ E
8. \((8)\) \(G(b) \Rightarrow K(a, b)\) \hspace{1cm} 7 \ \forall \ E
9. \((9)\) \(K(a, b)\) \hspace{1cm} 3, 8 \ MP
10. \((10)\) \((\forall y)(H(y) \Rightarrow \sim K(a, y))\) \hspace{1cm} 5, 6 \ MP
11. \((11)\) \(H(b) \Rightarrow \sim K(a, b)\) \hspace{1cm} 10 \ \forall \ E
12. \((12)\) \(\sim K(a, b)\) \hspace{1cm} 9 \ DN
13. \((13)\) \(\sim H(b)\) \hspace{1cm} 11, 12 \ MT
14. \((14)\) \(G(b) \Rightarrow \sim H(b)\) \hspace{1cm} 3, 13 \ CP
15. \((15)\) \(G(b) \Rightarrow \sim H(b)\) \hspace{1cm} 1, 4, 14 \ \exists \ E
16. \((16)\) \((\forall x)(G(x) \Rightarrow \sim H(x))\) \hspace{1cm} 15 \ \forall \ I
5. Consider first $U_1 = \mathcal{P}(\mathbb{Z}^+)$. If we take $X = \mathbb{Z}^+$ then, for all $Y \in U_1$,
\[
X \cap Y = \mathbb{Z}^+ \cap Y = Y = Y \cap \mathbb{Z}^+ = Y \cap X ,
\]
so that (a) holds. Of course (b) follows from (a), so (b) also holds. If we take $X = \emptyset$ then, for all $Y \in U_1$,
\[
X \cup Y = \emptyset \cup Y = Y = Y \cup \emptyset = Y \cup X ,
\]
so that (c) holds. Of course (d) follows from (c), so (c) also holds.

Consider now $U_2 = \mathcal{P}_{f}(\mathbb{Z}^+)$. Since the empty set is an element of $U_2$, (c) and (d) hold as for $U_1$. For each $Y \in U_2$, we can take $X = Y$ and then
\[
X \cap Y = Y \cap Y = Y = Y \cap Y = Y \cap X ,
\]
so that (b) holds. However (a) does not hold. To see this, suppose to the contrary that $X \in U_2$ exists such that the condition of (a) holds. Because $X$ is finite it will have some largest element $m$, so $m + 1 \notin X$. Let $Y = X \cup \{m + 1\}$. Then $Y \in U_2$ and
\[
X \cap Y = X \cap (X \cup \{m + 1\}) = X \neq Y ,
\]
which contradicts the condition of (a). This proves (a) does not hold.

Consider finally $U_3 = U_1 \setminus U_2$. Since $\mathbb{Z}^+$ is an element of $U_3$, (a) and (b) hold as for $U_1$. For each $Y \in U_3$, we can take $X = Y$ and then
\[
X \cup Y = Y \cup Y = Y = Y \cup Y = Y \cup X ,
\]
so that (d) holds. However (c) does not hold. To see this, suppose to the contrary that $X \in U_3$ exists such that the condition of (c) holds. Because $X$ is infinite we may choose any element $x \in X$ and put $Y = X \setminus \{x\}$, which is still infinite, so $Y \in U_3$. However,
\[
X \cup Y = X \cup (X \setminus \{x\}) = X \neq Y ,
\]
which contradicts the condition of (c). This proves (c) does not hold.
6. If \( W \) is a propositional variable then \( W^* \) becomes simply \( \sim W \), and certainly \( W^* \equiv \sim W \), which starts an induction. Suppose that \( W \) is not a propositional variable, and assume as inductive hypothesis that the conclusion holds for all wffs of length smaller than \( W \). Either \( W \) is \( \sim U \), \( U \land V \) or \( U \lor V \) for some wffs \( U, V \) of length smaller than \( W \). By the inductive hypothesis,

\[
U^* \equiv \sim U \quad \text{and} \quad V^* \equiv \sim V .
\]

If \( W = \sim U \) then

\[
W^* = (\sim U^*) \equiv (\sim U) = \sim W .
\]

If \( W = U \land V \) then, by one of de Morgan’s Laws,

\[
W^* = (U^* \lor V^*) \equiv (\sim U) \lor (\sim V) \equiv \sim (U \land V) = \sim W .
\]

If \( W = U \lor V \) then, by another of de Morgan’s Laws,

\[
W^* = (U^* \land V^*) \equiv (\sim U) \land (\sim V) \equiv \sim (U \lor V) = \sim W .
\]

In all cases \( W^* \equiv W \), and the result now follows by induction.

(6 marks)

7. Let \( W = W(X_1, \ldots, X_n) \) be a wff built from propositional variables \( X_1, \ldots, X_n \) using \( \sim, \land \) and \( \lor \). Put \( Y_1 = \sim X_1, \ldots, Y_n = \sim X_n \) and define

\[
\overline{W} = W(Y_1, \ldots, Y_n) .
\]

Observe that \( W^* = \overline{W}^* \). Consider any such wffs \( W_1 = W_1(X_1, \ldots, X_n) \) and \( W_2 = W_2(X_1, \ldots, X_n) \). We claim that

\[
W_1 \equiv W_2 \quad \text{if and only if} \quad \overline{W_1} \equiv \overline{W_2} . \tag{*}
\]

To see this, first suppose that \( W_1 \equiv W_2 \). Choose \( V(X_1), \ldots, V(X_n) \) from \( \{T, F\} \) and suppose that \( V(\overline{W_1}) = V(W_1(Y_1, \ldots, Y_n)) = T \). But \( V(Y_1), \ldots, V(Y_n) \) are also from \( \{T, F\} \), and, regarding \( Y_1, \ldots, Y_n \) temporarily as propositional variables, we have \( V(W_2(Y_1, \ldots, Y_n)) = T \) (since we are supposing \( W_1 \equiv W_2 \)).

Hence \( V(\overline{W_2}) = V(W_2(Y_1, \ldots, Y_n)) = T \). We have proved half of the logical equivalence \( \overline{W_1} \equiv \overline{W_2} \). Interchanging the roles of \( W_1 \) and \( W_2 \) in this argument proves the other half. Now suppose \( \overline{W_1} \equiv \overline{W_2} \). By what we have just proved

\[
W_1 = \overline{W_1} \equiv \overline{W_2} \equiv W_2 ,
\]

which completes the proof of our claim \( * \). By \( * \), with \( W_1^+ \) and \( W_2^+ \) in place of \( W_1 \) and \( W_2 \), and by the result of the previous exercise, we have

\[
W_1 \equiv W_2 \quad \text{if and only if} \quad \sim W_1 \equiv \sim W_2 , \quad \text{if and only if} \quad W_1^* \equiv W_2^* ,
\]

\[
\text{if and only if} \quad \overline{W_1} \equiv \overline{W_2} , \quad \text{if and only if} \quad W_1^+ \equiv W_2^+ ,
\]

and the full result is proved, since “if and only if” is transitive.

(8 marks)