1. Denote the number of symbols in a wff $W$ by $\#W$. If $P$ is a propositional variable then

$$\#P = 1, \quad \#(\sim P) = 4, \quad \#(P \lor P) = 5, \quad \#(\sim (P \lor P)) = 7,$$

$$\#(\sim (P \lor P)) = 8, \quad \#((P \lor P) \lor P) = 9.$$ 

If $W$ is any wff then $\#(\sim W) = \#W + 3$, so that, iterating this construction of adding a negation symbol and a pair of brackets, we may produce a wff $V$ such that

$$\#V = \#W + 3k$$

for any positive integer $k$. Since all positive integers $n \geq 7$ differ from 7, 8 and 9 by a multiple of three, we may apply this construction to the wffs with 7, 8 or 9 symbols to create a wff with $n$ symbols.

(5 marks)

It remains to prove that no wff exists having exactly 2, 3 or 6 symbols. Suppose a wff $W$ exists such that $\#W = 2, 3$ or 6. Certainly $W$ is not a propositional variable, so either

(i) $W = (\sim V)$ or (ii) $W = (V * U)$

for some wffs $V, W$ of smaller length, and where $*$ is one of the binary connectives. But then $\#W \geq 4$, ruling out any possibility that $\#W = 2$ or 3. Hence $\#W = 6$. In case (i), $\#V = 3$. In case (ii) $\#V + \#U = 3$, so that either $\#V = 2$ or $\#U = 2$. In either case we have produced a wff containing exactly 2 or 3 symbols, yielding a contradiction with what we have just observed with $V$ or $U$ in the role of $W$. Hence no such wff $W$ exists.

(3 marks)

2. (a) The truth table:

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \lor Q$</th>
<th>$P \land Q$</th>
<th>$(P \lor Q) \Rightarrow (P \land Q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$T$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$T$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

Counterexamples: (i) $P = T, Q = F$, and (ii) $P = F, Q = T$.

(3 marks)
(b) The truth table:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P \implies Q</th>
<th>Q \implies P</th>
<th>(P \implies Q) \implies (Q \implies P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Just one counterexample: \( P = F, Q = T \).

(3 marks)

3. (a) Claim: \((P \implies Q) \land (P \implies R) \vdash P \implies (Q \land R)\)

Proof:

1. (1) \((P \implies Q) \land (P \implies R)\) A
2. (2) \(P\) A
1. (3) \(P \implies Q\) 1 \land E
1. (4) \(P \implies R\) 1 \land E
1, 2 (5) \(Q\) 2, 3 MP
1, 2 (6) \(R\) 2, 4 MP
1, 2 (7) \(Q \land R\) 5, 6 \land I
1. (8) \(P \implies (Q \land R)\) 2, 7 CP

(4 marks)

(b) Claim: \((P \implies R) \land (Q \implies R) \vdash (P \lor Q) \implies R\)

Proof:

1. (1) \((P \implies R) \land (Q \implies R)\) A
2. (2) \(P \lor Q\) A
3. (3) \(P\) A
1. (4) \(P \implies R\) 1 \land E
1, 3 (5) \(R\) 3, 4 MP
6. (6) \(Q\) A
1. (7) \(Q \implies R\) 1 \land E
1, 6 (8) \(R\) 6, 7 MP
1, 2 (9) \(R\) 2, 3, 5, 6, 8 \lor E
1. (10) \((P \lor Q) \implies R\) 2, 9 CP

(4 marks)
(c) **Claim:** \( P \Rightarrow Q, \ R \Rightarrow S \vdash (P \lor R) \Rightarrow (Q \lor S) \)

**Proof:**

\[
\begin{array}{l}
1 \quad (1) \quad P \Rightarrow Q \quad A \\
2 \quad (2) \quad R \Rightarrow S \quad A \\
3 \quad (3) \quad P \lor R \quad A \\
4 \quad (4) \quad P \quad A \\
1,4 \quad (5) \quad Q \quad 1,4 \text{ MP} \\
1,4 \quad (6) \quad Q \lor S \quad 5 \lor I \\
7 \quad (7) \quad R \quad A \\
2,7 \quad (8) \quad S \quad 2,7 \text{ MP} \\
2,7 \quad (9) \quad Q \lor S \quad 8 \lor I \\
1,2,3 \quad (10) \quad Q \lor S \quad 3,4,6,7,9 \lor E \\
1,2 \quad (12) \quad (P \lor R) \Rightarrow (Q \lor S) \quad 3,10 \text{ CP} \\
\end{array}
\]

(4 marks)

4. **Claim:** \( \sim Q, \ P \Rightarrow Q \vdash \sim P \)

**Proof:**

\[
\begin{array}{l}
1 \quad (1) \quad \sim Q \quad A \\
2 \quad (2) \quad P \Rightarrow Q \quad A \\
3 \quad (3) \quad P \quad A \\
2,3 \quad (4) \quad Q \quad 2,3 \text{ MP} \\
1,2,3 \quad (5) \quad Q \land \sim Q \quad 1,4 \land I \\
1,2 \quad (6) \quad \sim P \quad 3,5 \text{ RAA} \\
\end{array}
\]

(4 marks)

5. (a) Working backwards

\[
(\sim P \land (P \Rightarrow Q)) \Rightarrow \sim Q
\]

\[
\begin{array}{cccccccc}
T & F & T & F & T & F & F & T \\
5 & 6 & 2 & 8 & 7 & 9 & 1 & 3 & 4
\end{array}
\]

produces a counterexample \( P = F, \ Q = T \).

(3 marks)

(b) Working backwards

\[
(\sim Q \land (P \Rightarrow Q)) \Rightarrow \sim P
\]

\[
\begin{array}{cccccccc}
T & F & T & T & F & F & F & T \\
5 & 6 & 2 & 8 & 7 & 9 & 1 & 3 & 4
\end{array}
\]

produces a contradiction, between assignations 7, 8, 9. In this case we can find a formal proof:
Claim: \((\sim Q \land (P \Rightarrow Q)) \Rightarrow \sim P\)

Proof:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(\sim Q \land (P \Rightarrow Q))</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>(\sim Q)</td>
<td>1 &amp; E</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>(P \Rightarrow Q)</td>
<td>1 &amp; E</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>(\sim P)</td>
<td>2, 3 MT</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>((\sim Q \land (P \Rightarrow Q))) (\Rightarrow \sim P)</td>
<td>1, 4 CP</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(5 marks)

6. (a) Claim: \((g \circ f \text{ injective}) \Rightarrow (f \text{ injective})\)

Proof: Suppose that \(g \circ f\) is injective. Let \(a_1, a_2 \in A\) and suppose that \(f(a_1) = f(a_2)\). Then

\[ g \circ f(a_1) = g(f(a_1)) = g(f(a_2)) = g \circ f(a_2). \]

Hence \(a_1 = a_2\), since \(g \circ f\) is injective. This verifies that \(f\) is injective.

(2 marks)

(b) Claim: \((g \circ f \text{ injective}) \Leftrightarrow (g \text{ injective})\)

Counterexample: Let \(f : \{0\} \to \mathbb{R}\) where \(f(0) = 0\), and \(g : \mathbb{R} \to \{0\}\) where \(g(x) = 0\) for all \(x \in \mathbb{R}\). Then \(g \circ f : \{0\} \to \{0\}\) where \(g \circ f(0) = 0\). Clearly \(g \circ f\) is injective, but \(g\) is not (because, for example, \(g(0) = 0 = g(1)\) but \(0 \neq 1\)).

(2 marks)

(c) Claim: \((g \circ f \text{ surjective}) \not\Rightarrow (f \text{ surjective})\)

Counterexample: Let \(f : \{0\} \to \mathbb{R}\) where \(f(0) = 0\), and \(g : \mathbb{R} \to \{0\}\) where \(g(x) = 0\) for all \(x \in \mathbb{R}\). Then \(g \circ f : \{0\} \to \{0\}\) where \(g \circ f(0) = 0\). Clearly \(g \circ f\) is surjective, but \(f\) is not (because, for example, \(1 \in \mathbb{R}\) and \(1\) is not a value of \(f\)).

(2 marks)

(d) Claim: \((g \circ f \text{ surjective}) \Rightarrow (g \text{ surjective})\)

Proof: Suppose that \(g \circ f\) is surjective and let \(c \in C\). Then \(c = g \circ f(a)\) for some \(a \in A\). Put \(b = f(a)\). Then \(b \in B\) and

\[ g(b) = g(f(a)) = g \circ f(a) = c. \]

This verifies that \(g\) is surjective.

(2 marks)
7. Write $X = \{x_1, x_2, \ldots, x_n\}$. Define $f : \mathcal{P}(X) \to S$ by the following rule, for each $Y \in \mathcal{P}(X)$:

$$f(Y) = 0.d_1d_2\ldots d_n$$

where, for each $i = 1, \ldots, n$,

$$d_i = \begin{cases} 1 & \text{if } x_i \in Y \\ 0 & \text{if } x_i \not\in Y. \end{cases}$$

(2 marks)

Suppose $Y_1, Y_2 \in \mathcal{P}(X)$ such that $f(Y_1) = 0.d_1d_2\ldots d_n = f(Y_2)$. If $y \in Y_1$ then $y = x_k$ for some $k$, so, by the rule for $f$, we have $d_i = 1$, which in turn yields $y \in Y_2$. This shows $Y_1 \subseteq Y_2$. Similarly $Y_2 \subseteq Y_1$, so $Y_1 = Y_2$. This proves that $f$ is injective.

(2 marks)

Suppose $\alpha = d_1d_2\ldots d_n \in S$ for some digits $d_1, d_2, \ldots, d_n \in \{0, 1\}$. Put

$$X = \{x_i \in X \mid d_i = 1\}.$$  

The rule for $f$ gives that $f(X) = \alpha$, which proves that $f$ is surjective.

(2 marks)

8. (a) Define $g : \mathcal{P}(\mathbb{Z}^+) \to T$ by the following rule, for each $Y \in \mathcal{P}(\mathbb{Z}^+)$:

$$f(Y) = 0.d_1d_2d_3\ldots$$

where, for each $i \in \mathbb{Z}^+$, $d_i = \begin{cases} 1 & \text{if } i \in Y \\ 0 & \text{if } i \not\in Y. \end{cases}$

(2 marks)

(b) Observe that $\mathcal{P}(\mathbb{Z}^+) = X \cup Y$ where

$$X = \{\text{infinite subsets of } \mathbb{Z}^+\} \cup \{\emptyset\}$$

and

$$Y = \{\text{nonempty finite subsets of } \mathbb{Z}^+\}.$$  

Define $h : X \to T$ by exactly the same rule as $g$ in the previous part, except that we interpret elements of $T$ using the base 2 expansion of real numbers. Any finite base 2 expansion of a nonzero real in $[0, 1]$ can be replaced by an expansion with infinitely many 1’s (by replacing the string 1000\ldots by the string 0111\ldots), and of course $h(\emptyset) = 0$, so $T = [0, 1]$. Hence $h$ is a bijection. It suffices to find a bijection $k : \mathcal{P}(\mathbb{Z}^+) \to X$ and form the composite $h \circ k$. From lectures $Y$ is countable, so there is a bijection $\ell : \mathbb{Z}^+ \to Y$, and then the following rule for $k$ finishes off the job:

$$k(\alpha) = \begin{cases} \alpha & \text{if } \alpha \not\in \{\mathbb{Z}^+ \setminus \{i\} \mid i \in \mathbb{Z}^+\} \cup Y \\ \mathbb{Z}^+ \setminus \{2i\} & \text{if } \alpha = \mathbb{Z}^+ \setminus \{i\} \text{ for some } i \in \mathbb{Z}^+ \\ \mathbb{Z}^+ \setminus \{2i - 1\} & \text{if } \alpha = \ell(i) \text{ for some } i \in \mathbb{Z}^+. \end{cases}$$

(6 marks)