Starred questions are suitable for students aiming for a credit or higher.

1. Decide if the following statements can have a well-defined truth-value:
   (a) The integer 57 is a prime.
   (b) The integer 57 is a Grothendieck prime.
   (c) London is in England.
   (d) This sentence is false.

2. State the converse and contrapositive of each of the following implications:
   (a) If the sun is shining we are heading off to the surf.
   (b) If I have listened attentively I will do well in the exam.
   *(c) It is necessary to have a valid password to log on to the computer.
   *(d) Only if my attendance is recorded will I be bothered coming to tutorials.

3. Use truth tables to verify that the following are theorems:
   (a) \[ P \land (P \Rightarrow Q) \Rightarrow Q \]
   (b) \[ \sim Q \land (P \Rightarrow Q) \Rightarrow \sim P \]
   *(c) \[ (P \Rightarrow Q) \land (Q \Rightarrow R) \Rightarrow (P \Rightarrow R) \]
   *(d) \[ (P \lor Q) \land (\sim P \lor R) \Rightarrow (Q \lor R) \]

4. Construct one truth table for each of the following compound propositions.
   (a) \( (P \lor Q) \lor R \)
   (b) \( (P \land Q) \lor R \)
   (c) \( (P \Rightarrow Q) \land R \)
   (d) \( (P \land Q) \Leftrightarrow R \)

From your table verify that \((c) \Rightarrow (b) \Rightarrow (a)\), but that none of these implications is reversible. Does \((d)\) imply any of \((a)\), \((b)\) or \((c)\)? Is \((d)\) implied by any of \((a)\), \((b)\) or \((c)\)?

5. Express the following true statements about integers using quantifiers and usual mathematical symbols:
   (a) The product of two negative integers is positive.
   (b) The difference of two negative integers need not be positive.
   (c) The set \( \mathbb{Z} \) has no largest or smallest element.
   (d) Any set consisting of negative integers has a largest element.
6. Rewrite the following formal statements so that no use of negation precedes a quantifier, and simplify if possible:

(a) \( \sim \left( \forall x \right) \left( \forall y \right) P(x, y) \)
(b) \( \sim \left( \forall x \right) \left( \exists y \right) \sim P(x, y) \lor \sim Q(x, y) \)

*(c) \( \sim \left( \sim \left( \exists x \right) \sim P(x) \right) \implies \left( \forall y \right) \left[ Q(x, y) \implies \sim R(x, y) \right] \)

*7. Interpret the following statements in simple words, and determine whether they are true when the universe of discourse is \( \mathbb{Z} \), \( \mathbb{R} \), \( \mathbb{R}^+ \), \( \mathbb{C} \), \( \mathbb{Z}_7 \) or \( \mathbb{Z}_8 \).

(a) \( \left( \forall x \right) \left( \exists y \right) x^2 = y \)
(b) \( \left( \forall x \right) \left( \exists y \right) x = y^2 \)
(c) \( \sim \left( \left( \forall x \right) \left( \exists y \right) x = 2y \right) \)
(d) \( \left( \forall x \right) \left[ x \neq 0 \implies \left( \exists y \right) xy = 1 \right] \)
(e) \( \left( \exists x \right) \left( \exists y \right) x \neq 0 \land y \neq 0 \land xy = 0 \)

*8. (due to Backhouse) Determine the validity of the following argument:

If Superman were able and willing to prevent evil, he would do so. If Superman were unable to prevent evil, he would be impotent; if he were unwilling to prevent evil, he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore Superman does not exist.