1. No, the statement is not a well-defined proposition because its truth value depends on the person reading the sentence and the relativistic nature of the word “calming”. Even if one fixed the person reading the sentence, there is a temporal aspect that might also change the truth value, depending on the person’s mood that can change over time.

2. (a) Let $P, Q, R$ denote the propositions “It’s bad”, “It’s depressing” and “Lectures are structured” respectively, so the hypotheses become

$$Q \Rightarrow P, \quad \sim R \Rightarrow Q,$$

and the conclusion becomes

$$R \Rightarrow \sim P.$$

The conclusion does not follow from the hypotheses. A counterexample would be to set $R = T, P = T$ and $Q = T$.

(b) Let $P, Q, R, S$ denote the propositions “It’s bad”, “It’s depressing”, “We live in Melbourne” and “We live in Sydney” respectively. The use of either-or in the first hypothesis is intended to be exclusive-or, so the hypotheses become

$$R \lor S, \quad \sim (R \land S), \quad Q \Rightarrow P, \quad R \Rightarrow Q,$$

and the conclusion becomes

$$S \Rightarrow \sim P.$$

The conclusion does not follow from the hypotheses. A counterexample would be to set $R = F, S = T, P = T$ and $Q = T$.

3. (a) If $P$ if false then $\sim P$ is true, so that $P \Rightarrow \sim P$ is true, so that

$$(P \Rightarrow \sim P) \Rightarrow P$$

is false, and we have produced a counterexample. The given statement therefore is not a theorem.

(b) Working backwards, by supposing the given statement is false, we get

$$(P \Rightarrow \sim P) \Rightarrow \sim P$$

and the assignations $5, 2, 7$ contradict the truth table for $\Rightarrow$. Thus the given statement is a theorem.
(c) Working backwards, by supposing the given statement is false, we get

\[(P \Rightarrow Q) \land (R \Rightarrow P) \Rightarrow (\sim Q \Rightarrow \sim R)\]

\[
\begin{array}{cccccccc}
T & T & F & T & T & T & T & F \\
13 & 10 & 8 & 2 & 9 & 11 & 12 & 1 \\
4 & 6 & 3 & 5 & 7
\end{array}
\]

and the assignations 13, 10, 8 contradict the truth table for \(\Rightarrow\). Thus the given statement is a theorem.

(d) If we let \(P\) be either true or false, \(Q\) be true and \(R\) be false, then \(\sim R\) is true and \(\sim Q\) is false, so that

\(\sim R \Rightarrow \sim Q\)

is false, whilst both \(P \Rightarrow Q\) and \(R \Rightarrow P\) are true, so that

\[(P \Rightarrow Q) \land (R \Rightarrow P)\]

is true, so that finally

\[(P \Rightarrow Q) \land (R \Rightarrow P) \Rightarrow (\sim R \Rightarrow \sim Q)\]

is false, and we have produced a counterexample. The given statement therefore is not a theorem.

*(e) If we let \(P\) be true and \(Q\) be false then \(P \lor Q\) is true and \(Q \Rightarrow \sim P\) is true, so that

\[
[(P \lor Q) \land (Q \Rightarrow \sim P)] \Rightarrow Q
\]

is false, and we have produced a counterexample. The given statement therefore is not a theorem.

*(f) If we let \(P\) be false and \(Q\) be true then \(P \lor Q\) is true and \(Q \Rightarrow \sim P\) is true, so that

\[
[(P \lor Q) \land (Q \Rightarrow \sim P)] \Rightarrow P
\]

is false, and we have produced a counterexample. The given statement therefore is not a theorem.

*(g) Working backwards, by supposing the given statement is false, we get

\[(P \lor Q) \land (\sim Q \Rightarrow \sim P) \Rightarrow Q\]

\[
\begin{array}{cccccccc}
T & T & F & T & F & T & T & F \\
11 & 4 & 6 & 2 & 8 & 7 & 5 & 9 \\
10 & 1 & 3
\end{array}
\]

and the assignations 11 and 10 give a contradiction for \(P\). Thus the given statement is a theorem.

4. (a) \(f : x \mapsto x + 1\) for all \(x \in \mathbb{N}\).  
(b) \(g : x \mapsto x - 1\) for all \(x \in \mathbb{Z}^+\).

(c) \(h : x \mapsto \begin{cases} 
0 & \text{if } x = 0 \\
2x & \text{if } x > 0 \\
-2x - 1 & \text{if } x < 0
\end{cases}\)

(d) \(k : x \mapsto \begin{cases} 
0 & \text{if } x = 1 \\
x/2 & \text{if } x \text{ is even} \\
-(x - 1)/2 & \text{if } x \geq 3 \text{ is odd}
\end{cases}\)
5. Define \( g : \mathbb{Z}^+ \rightarrow \mathbb{Q} \) by the rule
\[
g(x) = \begin{cases} 
0 & \text{if } x = 1 \\
f(x/2) & \text{if } x \text{ is even} \\
-f((x-1)/2) & \text{if } x \geq 3 \text{ is odd}
\end{cases}
\]

*6. Suppose that \( X \) is infinite, so there is no bijection between \( X \) and \([n]\) for each natural number \( n \). Certainly \( X \) is nonempty, since there is no bijection between \( X \) and \( \emptyset \). Hence we may choose \( f(1) \in X \). If \( X = \{f(1)\} \) then there is a bijection between \( X \) and \([1]\), namely \( f(1) \mapsto 1 \), a contradiction. Hence \( X \neq \{f(1)\} \), so we can choose \( f(2) \in X \setminus \{f(1)\} \), so that
\[
f(1), f(2)
\]
is a list of two distinct elements from \( X \). Suppose, as inductive hypothesis, that we have a list of \( n \) distinct elements
\[
f(1), f(2), \ldots, f(n)
\]
from \( X \), where \( n \) is a positive integer. If \( X = \{f(1), f(2), \ldots, f(n)\} \) then we have a bijection from \( X \) to \([n]\), namely
\[
f(1) \mapsto 1, f(2) \mapsto 2, \ldots, f(n) \mapsto n,
\]
a contradiction. Hence \( X \neq \{f(1), f(2), \ldots, f(n)\} \), so we may choose \( f(n+1) \in X \setminus \{f(1), f(2), \ldots, f(n)\} \).

By induction we have created a function \( f : \mathbb{Z}^+ \rightarrow X \) with the property that, for each positive integer \( n \), \( f(n) \) is different from each of \( f(1), \ldots, f(n-1) \). This shows \( f \) is injective.

*7. Suppose that \( X \) is infinite and \( Y \) is finite, so there is a bijection \( g : Y \rightarrow [n] \) for some natural number \( n \). Without any loss of generality, we may suppose \( X \) and \( Y \) are disjoint (for otherwise we may replace \( Y \) by a subset, which is finite and disjoint from \( X \)). If \( n = 0 \) then \( Y \) is empty, so the identity mapping is a bijection between \( X \) and \( X \cup Y = X \). So we may suppose \( n > 0 \). Let \( f : \mathbb{Z}^+ \rightarrow X \) be the injective mapping found in the previous exercise. Then the mapping \( h : X \cup Y \rightarrow X \) given by the rule
\[
h(x) = \begin{cases} 
x & \text{if } x \in X \setminus \text{im} f \\
f(m+n) & \text{if } x = f(m) \text{ for some } m \in \mathbb{Z}^+ \\
f(g(y)) & \text{if } y \in Y
\end{cases}
\]
is clearly a bijection.

*8. The mapping \( f : (-\pi/2, \pi/2) \rightarrow \mathbb{R} \) where \( f(x) = \tan x \) is clearly a bijection. It follows quickly that the mapping \( g : (\alpha, \beta) \rightarrow \mathbb{R} \) defined by
\[
g(x) = \tan \left( \frac{\pi(x-\alpha)}{\beta-\alpha} - \frac{\pi}{2} \right)
\]
is a bijection. By the previous exercise, there is a bijection $h : [\alpha, \beta] \to (\alpha, \beta)$, so that $g \circ h : [\alpha, \beta] \to \mathbb{R}$ is also a bijection. If $[\alpha, \beta]$ is countable then there is a bijection $k : \mathbb{Z}^+ \to [\alpha, \beta]$, so that $g \circ h \circ k : \mathbb{Z}^+ \to \mathbb{R}$ is a bijection, contradicting that $\mathbb{R}$ is not countable. Hence $[\alpha, \beta]$ is not countable.

*9. By a simple induction, it suffices to check that if $X$ and $Y$ are two countable sets then $X \cup Y$ is countable. We may suppose $X$ and $Y$ are disjoint, for otherwise we could replace $Y$ by a subset which is disjoint from $X$. The result is clear if both sets are finite or (by an earlier exercise) if one is infinite and the other finite. It suffices then to assume that $X$ and $Y$ are both infinite. Then there are bijections $f : \mathbb{Z}^+ \to X$ and $g : \mathbb{Z}^+ \to Y$. Clearly the map $h : \mathbb{Z}^+ \to X \cup Y$ defined by

$$h(x) = \begin{cases} f(x/2) & \text{if } x \text{ is even} \\ g((x-1)/2) & \text{if } x \text{ is odd} \end{cases}$$

is a bijection, which proves $X \cup Y$ is countable, and we are done.

*10. Suppose that $\mathbb{R} \setminus \mathbb{Q}$ is countable. We know from lectures that $\mathbb{Q}$ is countable. By the previous exercise,

$$\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q}$$

is countable. But this contradicts Cantor’s Theorem that says $\mathbb{R}$ is uncountable. Hence $\mathbb{R} \setminus \mathbb{Q}$ is uncountable.