This assignment comprises a total of 60 marks, and is worth 15% of the overall assessment. It should be completed, accompanied by a signed cover sheet, and handed in at the lecture on Monday 22 April. Acknowledge any sources or assistance.

1. Construct truth tables for each of (a) and (b), and find all counterexamples in each case.

(a) \((Q \Rightarrow P) \Rightarrow (P \Rightarrow Q)\) 
(b) \((Q \lor P) \Rightarrow (Q \land P)\)

(6 marks)

2. Use the rules of deduction in the Propositional Calculus to find formal proofs for the following sequents:

(a) \((R \Rightarrow P) \land (R \Rightarrow Q) \vdash R \Rightarrow (P \land Q)\) 
(b) \((Q \Rightarrow R) \land (P \Rightarrow R) \vdash (Q \lor P) \Rightarrow R\) 
(c) \(P \Rightarrow R, Q \Rightarrow S \vdash (P \lor Q) \Rightarrow (R \lor S)\)

(12 marks)

3. Find a formal proof in the Propositional Calculus for the sequent

\(\sim R, Q \Rightarrow R \vdash \sim Q\)

without using Modus Tollens (MT). (This shows that MT may be deleted from the rules of deduction without compromising provability.)

(4 marks)

4. Use truth values to determine which one of the following is a theorem (in the sense of always being true).

(a) \(\left(Q \land (\sim P \Rightarrow \sim Q)\right) \Rightarrow P\) 
(b) \(\left(P \land (\sim P \Rightarrow \sim Q)\right) \Rightarrow Q\)

For the one that isn’t a theorem, produce a counterexample. For the one that is a theorem, provide a formal proof also using rules of deduction in the Propositional Calculus.

(8 marks)
5. Consider well-formed formulae $W_n$, for each positive integer $n$, defined as follows, where $P_1, P_2, \ldots$ are propositional variables:

\[ W_1 = P_1 \quad \text{and} \quad W_{k+1} = (W_k \leftrightarrow P_{k+1}) \quad \text{for each} \ k \geq 1. \]

(a) Write out $W_2$, $W_3$ and $W_4$ explicitly in terms of $P_1$, $P_2$, $P_3$ and $P_4$.

(b) Prove that $W_n$ contains $4n - 3$ symbols, for each $n$, where a propositional variable (including its subscript) counts as one symbol.

(c) Prove that $W_n$ has truth value $T$ if and only if an even number of the propositional variables $P_1,\ldots,P_n$ have truth value $F$, for each $n$.

(15 marks)

6. Recall that if $X$ is a set then $\mathcal{P}(X)$ is the set of all subsets of $X$, called the power set of $X$.

(a) Suppose that $A$ and $B$ are sets and that there is a surjective function $h : A \to B$. Prove carefully that there exists an injective function $g : B \to A$.

(b) For any real number $x \in [0,1]$, denote its decimal expansion by

\[ x = 0.d_1d_2\ldots d_n\ldots \]

where each $d_i$ is a digit from 0 to 9. The following rules for functions $f, h : [0,1] \to \mathcal{P}(\mathbb{Z}^+)$ are not quite well-defined:

\[ f(x) = \{ d_i \times 10^i \mid d_i \neq 0 \} \quad \text{and} \quad h(x) = \{ i \in \mathbb{Z}^+ \mid d_i = 0 \}. \]

Explain how to modify them so that both $f$ and $h$ become well-defined. Prove carefully that $f$ then becomes injective and that $h$ then becomes surjective.

(c) Use parts (a) and (b) and the Schröder-Bernstein Theorem to deduce that the interval $[0,1]$ and the power set $\mathcal{P}(\mathbb{Z}^+)$ have the same cardinality.

(15 marks)