Starred questions are suitable for students aiming for a credit or higher.

1. Let $\phi : R \rightarrow S$ be a ring homomorphism. Prove that $\phi$ is one-one if and only if $\ker \phi = \{0\}$.

2. Let $I$ be an ideal of a ring $R$ and define the natural map $\nu : R \rightarrow R/I$ by $x\nu = I + x$ for all $x \in R$. Verify that $\nu$ is an onto ring homomorphism and $\ker \phi = I$.

3. Let $R$ and $S$ be rings and define the direct sum of $R$ and $S$ to be $R \oplus S = \{(x, y) \mid x \in R, \ y \in S\}$.

   (a) Verify that $R \oplus S$ becomes a ring with respect to coordinatewise operations.

   (b) Prove that if $R \cong R'$ and $S \cong S'$ then $R \oplus S \cong R' \oplus S'$.

4. Suppose that $\phi_1 : R \rightarrow S_1$ and $\phi_2 : R \rightarrow S_2$ are ring homomorphisms. Define $\phi : R \rightarrow S_1 \oplus S_2$ by the rule $x\phi = (x\phi_1, x\phi_2)$ for $x \in R$.

   Verify that $\phi$ is a ring homomorphism and $\ker \phi = \ker \phi_1 \cap \ker \phi_2$.

*5. Prove that $\mathbb{Z}_m \oplus \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ if and only if $m$ and $n$ are coprime.

*6. Let $\phi : F \rightarrow S$ be a ring homomorphism where $F$ is a field. Show that $\phi$ is injective if and only if $1\phi \neq 0$.

*7. Let $\phi : \text{Mat}_2(\mathbb{R}) \rightarrow F$ be a ring homomorphism where $F$ is a field. Show that $\phi$ is the zero map.

8. Call an element $e$ of a ring idempotent if $e^2 = e$.

   (a) Prove that the only idempotents in an integral domain are 0 and 1.

   (b) Let $\phi : R \rightarrow F$ be a ring homomorphism where $F$ is a field and $R$ is a ring with identity. Verify that if $\phi$ is not the zero map then $1\phi = 1$.

9. Prove that the only nonzero ring homomorphism from $\mathbb{Q}$ to $\mathbb{R}$ is the identity map that embeds $\mathbb{Q}$ in $\mathbb{R}$ as a subring.

*10. Prove that if $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a ring homomorphism then either $\phi$ is the zero map or $\phi$ is the identity mapping.