1. Use Jensen’s Inequality and the fact that \(-\ln\) is a convex function to prove the arithmetic mean geometric mean inequality:

\[ \sqrt[m]{x_1 x_2 \cdots x_m} \leq \frac{1}{m} (x_1 + x_2 + \cdots + x_m) \]

for all positive numbers \(x_1, x_2, \ldots, x_m\). [Hint: consider a random variable taking \(m\) equally probable values \(x_i\).]

2. Suppose that \(f\) is convex on a convex subset \(C\) in \(\mathbb{R}^n\). Let \(a\) be a fixed real number. Show that the set of points \(x \in C\) with \(f(x) \leq a\) is a convex set.

3. Suppose that an investor has the choice of a safe investment, with a fixed return of \(r\) for every dollar invested, and a risky investment returning \(s\) with probability \(p\) or zero with probability \(1 - p\) for every dollar invested. Apply Corollary 8.16 of O’Brien’s book (done in Lecture 17) to determine the log-optimal portfolio. Assume that \(1 \leq r < s\) and \(0 < p < 1\) since otherwise the optimal strategy is obvious. [Hint: You should find two separate cases, corresponding to whether or not the fraction of funds allocated to the risky investment is zero or nonzero.]

4. Prove that the function \(f\) on \(\mathbb{R}^2\) defined by \(f(x, y) = x^2 + y^2\) is convex by finding, for each point \((x_0, y_0) \in \mathbb{R}^2\), a function \(h(x, y) = ax + by + c\) such that \(h(x_0, y_0) = f(x_0, y_0)\) and \(h(x, y) < f(x, y)\) for all other \((x, y) \in \mathbb{R}^2\). [Note that graph of \(h\) must be the tangent plane to the graph of \(f\) at \((x_0, y_0, f(x_0, y_0))\).]

5. There are 5 horses in a race and the tote prices are 10, 6, 4, 4 and 3. Verify that it is not possible to bet in a way that guarantees a positive return irrespective of the probabilities of the various horses winning. Now suppose that you know that the probabilities of the various horses winning are (in the same order as above) 0.1, 0.1, 0.2, 0.2, 0.4.

   (i) Compute \(\sum_j \min(q_j, \frac{b_j}{p_j/q_j})\) for each \(j\), and hence determine which horses to bet on.

   (ii) Compute \(b_0\), the fraction of available funds that should not be bet.

   (iii) Compute the nonzero \(b_i\) (corresponding to your answer to Part (i)).