Background: Section 4.4 – American Options in the CRR Model.

**Exercise 1** Assume the CRR model $\mathcal{M} = (B, S)$ with $T = 3$, the stock price $S_0 = 100$, $S_u^1 = 120$, $S_d^1 = 90$, and the risk-free interest rate $r = 0.1$. Consider the American put option on the stock $S$ with the maturity date $T = 3$ and the constant strike price $K = 121$.

(a) Find the arbitrage price $P^a_t$ of the American put option for $t = 0, 1, 2, 3$.

(b) Find the rational exercise times $\tau^*_t$, $t = 0, 1, 2, 3$ for the holder of the American put option.

(c) Show that there exists an arbitrage opportunity for the issuer if the option is not rationally exercised by its holder.

**Exercise 2** Assume the CRR model $\mathcal{M} = (B, S)$ with $T = 3$, the stock price $S_0 = 100$, $S_u^1 = 120$, $S_d^1 = 90$, and the risk-free interest rate $r = 0$. Consider the American call option with the expiration date $T = 3$ and the running payoff $g(S_t, t) = (S_t - K_t)^+$, where the variable strike price equals $K_0 = K_1 = 100$, $K_2 = 105$ and $K_3 = 110$.

(a) Find the arbitrage price $X^a_t$ of the American call option for $t = 0, 1, 2, 3$.

(b) Find the holder’s rational exercise times $\tau^*_0$ for the American call option.

(c) Find the issuer’s replicating strategy for the American call option up to the rational exercise time $\tau^*_0$.

**Exercise 3** Consider the CRR binomial lattice model $\mathcal{M} = (B, S)$ with the initial stock price $S_0 = 9$, the interest rate $r = 0.01$ and the volatility equals $\sigma = 0.1$ per annum. Use the CRR parametrization for $u$ and $d$, that is,

$$u = e^{\sigma \sqrt{\Delta t}}, \quad d = \frac{1}{u},$$

with the time increment $\Delta t = 1$. 

We consider call and put options with the expiration date $T = 5$ years and strike $K = 10$.

(a) Compute the price process $C_t$, $t = 0, 1, \ldots, 5$ of the European call option using the binomial lattice method.

(b) Compute the price process $P_t$, $t = 0, 1, \ldots, 5$ for the European put option.

(c) Does the put-call parity relationship hold for $t = 0$?

(d) Compute the price process $P^a_t$, $t = 0, 1, \ldots, 5$ for the American put option. Will the American put option be exercised before the expiration date $T = 5$ by its rational holder?

**Exercise 4** (MATH3975) Consider the game option (See Section 4.5) with the expiration date $T = 12$ and the payoff functions $h(S_t)$ and $\ell(S_t)$ where

$$H_t = h(S_t) = (K - S_t)^+ + \alpha$$

and

$$L_t = \ell(S_t) = (K - S_t)^+$$

where $\alpha = 0.02$ and $K = 27$. Assume the CRR model with $d = 0.9, u = 1.1, r = 0.05$ and $S_0 = 25$.

(a) Compute the arbitrage price process $(X^g_t)_{t=0}^T$ for the game option using the recursive formula, for $t = 0, 1, \ldots, T - 1$,

$$X^g_t = \min \left\{ h(S_t), \max \left[ \ell(S_t), (1 + r)^{-1} (\bar{p}X^g_{t+1} + (1 - \bar{p})X^g_{t+1}) \right] \right\}$$

with $\pi_T(X^g) = \ell(S_T)$.

(b) Find the optimal exercise times $\tau^*_0$ and $\sigma^*_0$ for the holder and the issuer of the game option. Recall that

$$\tau^*_0 = \inf \left\{ t \in \{0, 1, \ldots, T\} \mid X^g_t = \ell(S_t) \right\}$$

and

$$\sigma^*_0 = \inf \left\{ t \in \{0, 1, \ldots, T\} \mid X^g_t = h(S_t) \right\}.$$