Exercise 1 Consider the Black-Scholes model $\mathcal{M} = (B, S)$ with the initial stock price $S_0 = 9$, the continuously compounded interest rate $r = 0.01$ per annum and the stock price volatility equals $\sigma = 0.1$ per annum.

(a) Using the Black-Scholes call option pricing formula

$$C_0 = S_0 N(d_+(S_0, T)) - Ke^{-rT} N(d_-(S_0, T))$$

compute the price $C_0$ of the European call option with strike price $K = 10$ and maturity $T = 5$ years.

(b) Using the Black-Scholes put option pricing formula

$$P_0 = Ke^{-rT} N(-d_-(S_0, T)) - S_0 N(-d_+(S_0, T))$$

compute the price $P_0$ for the European put option with strike price $K = 10$ and maturity $T = 5$ years.

(c) Does the put-call parity relationship

$$C_0 - P_0 = S_0 - Ke^{-rT}$$

hold?

(d) Recompute the prices of call and put options for modified maturities $T = 5$ months and $T = 5$ days.

(e) Explain the observed pattern of call and put prices when the time to maturity goes to zero.
**Exercise 2** Assume that the stock price $S$ is governed under the martingale measure $\mathbb{P}$ by the Black-Scholes stochastic differential equation

$$dS_t = S_t \left( r \, dt + \sigma \, dW_t \right)$$

where $\sigma > 0$ is a constant volatility and $r$ is a constant short-term interest rate. Let $0 < L < K$ be real numbers. Consider the contingent claim with the payoff $X$ at maturity date $T > 0$ given as $X = \min(|S_T - K|, L)$.

(a) Sketch the profile of the payoff $X$ as the function of the stock price $S_T$ at maturity date $T$ and find the decomposition of the payoff $X$ in terms of the payoffs of standard call and put options with different strikes.

(b) Compute the arbitrage price $\pi_t(X)$ at any date $t \in [0, T]$. Take for granted the Black-Scholes pricing formulae for European call and put options.

(c) Find the limits of the arbitrage price $\lim_{L \to 0} \pi_0(X)$ and $\lim_{L \to \infty} \pi_0(X)$.

(d) Find the limit of the arbitrage price $\lim_{\sigma \to \infty} \pi_0(X)$.

**Exercise 3** We consider the call option pricing function, that is, the function $c : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ such that $C_t = c(S_t, t)$ for all $t \in [0, T]$ where $C_t$ is the Black-Scholes price of the call option.

(a) Check directly that the pricing function $c$ satisfies the Black-Scholes PDE. To this end, compute the partial derivatives $c_s, c_{ss}$ and $c_t$, which are given in Section 5.5 of the course notes.

(b) Show that $v$ satisfies the terminal condition $v(s, T) = (s - K)^+$ in the sense that $\lim_{t \to T} v(s, t) = (s - K)^+$.

**Exercise 4** (MATH3975) Consider the stock price process $S$ under the Black and Scholes assumption, that is,

$$S_t = S_0 \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right)$$

where $W$ is the Wiener process under the martingale measure $\mathbb{P}$.

(a) Show that $\tilde{S}_t := e^{-rt}S_t$ is a martingale under $\mathbb{P}$ with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ generated by the stock price process $S$. Hint: Use the property that $\tilde{S}_s$ is independent of $\mathcal{F}_s$ for $0 \leq s < t$.

(b) Compute the expectation $\mathbb{E}_\mathbb{P}(S_t)$ and the variance $\text{Var}_\mathbb{P}(S_t)$ of the stock price under the martingale measure $\mathbb{P}$ using the martingale property of $\tilde{S}$ under $\mathbb{P}$.