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New Features

- In the case of multi-period market models, we need to deal with two important new features.
- First, the investors can trade in assets at any specific date \( t \in \{0, 1, 2, \ldots, T\} \) where \( T \) is the horizon date.
- Second, the investors can gather information over time, since the fluctuations of asset prices can be observed.
- We will need to introduce the concept of a self-financing trading strategy.
- We have to determine how the level of information available to investors evolves over time.
- The latter aspect leads to the probabilistic concepts of \( \sigma \)-fields and filtrations.
Outline

We will examine the following issues:

1. Partitions and $\sigma$-fields.
2. Filtrations and adapted stochastic processes.
3. Conditional expectation with respect to a partition or a $\sigma$-field.
PART 1

PARTITIONS AND $\sigma$-FIELDS
The concept of a \( \sigma \)-field can be used to describe the amount of information available at a given moment.

Let \( \mathbb{N} = \{1, 2, \ldots \} \) be the set of all natural numbers.

**Definition (\( \sigma \)-Field)**

A collection \( \mathcal{F} \) of subsets of \( \Omega \) is called a \( \sigma \)-field (or a \( \sigma \)-algebra) whenever:

1. \( \Omega \in \mathcal{F} \),
2. if \( A \in \mathcal{F} \) then \( A^c := \Omega \setminus A \in \mathcal{F} \),
3. if \( A_i \in \mathcal{F} \) for all \( i \in \mathbb{N} \) then \( \bigcup_{i=1}^{\infty} A_i \in \mathcal{F} \).
Interpretation of a $\sigma$-Field

- The set of information has to contain all possible states, so that we postulate that $\Omega$ belongs to any $\sigma$-field.

- Any set $A \in \mathcal{F}$ is interpreted as an observed event.

- If an event $A \in \mathcal{F}$ is given, that is, some collection of states is given, then the remaining states can also be identified and thus the complement $A^c$ is also an event.

- The idea of a $\sigma$-field is to model a certain level of information.

- In particular, as the $\sigma$-field becomes larger, more and more events can be identified.

- We will later introduce a concept of an increasing flow of information, formally represented by an ordered (increasing) family of $\sigma$-fields.
Definition (Probability Measure)

A map $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ is called a **probability measure** if:

1. $\mathbb{P}(\Omega) = 1$,
2. for any sequence $A_i, i \in \mathbb{N}$ of pairwise disjoint events we have

\[
\mathbb{P}\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mathbb{P}(A_i).
\]

The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space**.

- By convention, the probability of all possibilities is 1 (see 1).
- Probability should satisfy $\sigma$-additivity (see 2)
- Note that $\mathbb{P}(\emptyset) = 0$ and for an arbitrary event $A \in \mathcal{F}$ we have $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$. 
Example: \( \sigma \)-Fields

Example (5.1)

We take \( \Omega = \{ \omega_1, \omega_2, \omega_3, \omega_4 \} \) and we define the \( \sigma \)-fields:

\[
\begin{align*}
\mathcal{F}_1 &= \{ \emptyset, \Omega \} \\
\mathcal{F}_2 &= \{ \emptyset, \Omega, \{ \omega_1, \omega_2 \}, \{ \omega_3 \}, \{ \omega_4 \}, \{ \omega_1, \omega_2, \omega_3 \}, \{ \omega_1, \omega_2, \omega_4 \}, \{ \omega_3, \omega_4 \} \} \\
\mathcal{F}_3 &= 2^{\Omega} \quad \text{(the class of all subsets of } \Omega).\
\end{align*}
\]

Note that \( \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \), that is, the information increases:

- \( \mathcal{F}_1 \): no information, except for the set \( \Omega \).
- \( \mathcal{F}_2 \): partial information, since we cannot distinguish between the occurrence of either \( \omega_1 \) or \( \omega_2 \).
- \( \mathcal{F}_3 \): full information, since \( \{ \omega_1 \}, \{ \omega_2 \}, \{ \omega_3 \} \) and \( \{ \omega_4 \} \) can be observed.
Example (5.1 Continued)

- We define the probability measure $\mathbb{P}$ on the $\sigma$-field $\mathcal{F}_2$

  \[ \mathbb{P}(\{\omega_1, \omega_2\}) = \frac{2}{3}, \quad \mathbb{P}(\{\omega_3\}) = \frac{1}{6}, \quad \mathbb{P}(\{\omega_4\}) = \frac{1}{6}. \]

- The $\sigma$-additivity of $\mathbb{P}$ leads to

  \[ \mathbb{P}(\{\omega_1, \omega_2\} \cup \{\omega_3\} \cup \{\omega_4\}) = 1 = \mathbb{P}(\Omega). \]

- Note that $\mathbb{P}$ is not yet defined on the $\sigma$-field $\mathcal{F}_3 = 2\Omega$ and in fact the extension of $\mathbb{P}$ from $\mathcal{F}_2$ to $\mathcal{F}_3$ is not unique.

- For any $\alpha \in [0, 2/3]$ we may set

  \[ \mathbb{P}_\alpha(\{\omega_1\}) = \alpha = \frac{2}{3} - \mathbb{P}_\alpha(\{\omega_2\}). \]
Definition

Let $I$ be some index set. Assume that we are given a collection $(B_i)_{i \in I}$ of subsets of $\Omega$. Then the smallest $\sigma$-field containing this collection is denoted by $\sigma((B_i)_{i \in I})$ and is called the $\sigma$-field generated by $(B_i)_{i \in I}$.

Definition (Partition)

By a partition of $\Omega$, we mean any collection $\mathcal{P} = (A_i)_{i \in I}$ of non-empty subsets of $\Omega$ such that the sets $A_i$ are pairwise disjoint, that is, $A_i \cap A_j = \emptyset$ whenever $i \neq j$ and $\bigcup_{i \in I} A_i = \Omega$.

Lemma

A partition $\mathcal{P} = (A_i)_{i \in I}$ generates a $\sigma$-field $\mathcal{F}$ if every set $A \in \mathcal{F}$ can be represented as follows: $A = \bigcup_{j \in J} A_j$ for some subset $J \subset I$. 
Partition Associated with a $\sigma$-Field

Definition (Partition Associated with $\mathcal{F}$)

A partition of $\Omega$ associated with a $\sigma$-field $\mathcal{F}$ is a collection of non-empty sets $A_i \in \mathcal{F}$ for some $i \in I$ such that

1. $\Omega = \bigcup_{i \in I} A_i$.
2. The sets $A_i$ are pairwise disjoint, i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$.
3. For each $A \in \mathcal{F}$ there exists $J \subseteq I$ such that $A = \bigcup_{i \in J} A_i$.

Lemma

For any $\sigma$-field $\mathcal{F}$ of subsets of a finite state space $\Omega$, a partition associated with this $\sigma$-field always exists and is unique.
Partition Associated with a $\sigma$-Field

Further properties of partitions:

- If $\Omega$ is **countable** then for any $\sigma$-field $\mathcal{F}$ there exists a unique partition $\mathcal{P}$ of $\Omega$ associated with $\mathcal{F}$. It is also clear that this partition generates $\mathcal{F}$, so that $\mathcal{F} = \sigma(\mathcal{P})$.

- The sets $A_i$ in a partition must be smallest, specifically, if $\mathcal{F} = \sigma(\mathcal{P})$ and $A \in \mathcal{F}$ is such that $A \subseteq A_i$ then $A = A_i$.

- The probability of any $A \in \mathcal{F}$ equals the sum of probabilities of $A_i$s in the partition generating $\mathcal{F}$, specifically,

$$A = \bigcup_{i \in J} A_i \quad \Rightarrow \quad \mathbb{P}(A) = \sum_{i \in J} \mathbb{P}(A_i).$$
Example (5.2)

- Consider the $\sigma$-field $\mathcal{F}_2$ introduced in Example 5.1.
- The unique partition associated with $\mathcal{F}_2$ is given by
  \[ P_2 = \left\{ \{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\} \right\}. \]
- Define the probabilities
  \[ P(\{\omega_1, \omega_2\}) = \frac{2}{3}, \quad P(\{\omega_3\}) = \frac{1}{6}, \quad P(\{\omega_4\}) = \frac{1}{6}. \]
- Then for each event $A \in \mathcal{F}_2$ the probability of $A$ can be easily evaluated, for instance
  \[ P(\{\omega_1, \omega_2, \omega_4\}) = P(\{\omega_1, \omega_2\}) + P(\{\omega_4\}) = \frac{5}{6}. \]
Let $\mathcal{F}$ be an arbitrary $\sigma$-field of subsets of $\Omega$.

In the next definition, we do not assume that the sample space is discrete.

**Definition ($\mathcal{F}$-Measurability)**

A map $X : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}$-measurable if for every closed interval $[a, b] \subset \mathbb{R}$ the preimage (i.e. the inverse image) under $X$ belongs to $\mathcal{F}$, that is,

$$X^{-1}([a, b]) := \{\omega \in \Omega \mid X(\omega) \in [a, b]\} \in \mathcal{F}.\]

Equivalently, for any real number $x$

$$X^{-1}((-\infty, x]) := \{\omega \in \Omega \mid X(\omega) \leq x\} \in \mathcal{F}.$$

If $X$ is $\mathcal{F}$-measurable then $X$ is called a **random variable** on $(\Omega, \mathcal{F})$. 
Proposition (5.1)

Let $X : \Omega \to \mathbb{R}$ be an arbitrary function. Let $\mathcal{F}$ be a $\sigma$-field and $\mathcal{P} = (A_i)_{i \in I}$ be the unique partition associated with $\mathcal{F}$. Then $X$ is $\mathcal{F}$-measurable if and only if for each $A_i \in \mathcal{P}$ there exists a constant $c_i$ such that $X$ maps $A_i$ to $\{c_i\}$, that is,

$$X(\omega) = c_i \text{ for all } \omega \in F_i.$$

Proof of Proposition 5.1 ($\Rightarrow$).

($\Rightarrow$) Assume that $(A_i)_{i \in I}$ is a partition of the $\sigma$-field $\mathcal{F}$ and that $X$ is $\mathcal{F}$-measurable. Let $j \in I$ be an index and let an element $\omega \in A_j$ be arbitrary. Define $c_j := X(\omega)$. We wish to show that $X(\omega) = c_j$ for all $\omega \in A_j$. Since $X$ is $\mathcal{F}$-measurable, $X^{-1}(c_j) \in \mathcal{F}$. □
Proof of Proposition 5.1

By properties 2. and 3. in the definition of a $\sigma$-field, we have

$$\emptyset \neq X^{-1}(c_j) \cap A_j \in \mathcal{F}.$$ 

It is obvious that the inclusion

$$X^{-1}(c_j) \cap A_j \subset A_j$$

holds. Therefore, from the aforementioned minimality property of the sets contained in the partition, we obtain

$$X^{-1}(c_j) \cap A_j = A_j.$$ 

But this means that $X(\omega) = c_j$ for all $\omega \in A_j$ and thus $X$ is constant on $A_j$. By varying $j$, we obtain such $c_j$'s for all $j \in I$. 

\[\square\]
Proof of Proposition 5.1 $(\Leftarrow)$. 

$(\Leftarrow)$ Assume now that $X : \Omega \to \mathbb{R}$ is a function, which is constant on all sets $A_j$ belonging to the partition and that the $c_j$ are given as in the statement of the proposition.

Let $[a, b]$ be a closed interval in $\mathbb{R}$. We define 

$$C := \{c_j \mid j \in I \text{ and } c_j \in [a, b]\}.$$ 

Since $\bigcup_{i \in I} A_i = \Omega$ no other elements then the $c_j$ occur as values of $X$. Therefore, 

$$X^{-1}([a, b]) = X^{-1}(C) = \bigcup_{j \mid c_j \in C} X^{-1}(c_j) = \bigcup_{j \mid c_j \in C} A_j \in \mathcal{F}$$

where the last equality holds by property 3. of a $\sigma$-field.
PART 2

FILTRATIONS AND ADAPTED STOCHASTIC PROCESSES
In a typical application, the information about random events increases over time.

To model the information flow, we use the concept of a filtration.

The following definition covers the cases of the discrete and continuous time.

**Definition (Filtration)**

A family \((\mathcal{F}_t)_{0 \leq t \leq T}\) of \(\sigma\)-fields on \(\Omega\) is called a filtration if \(\mathcal{F}_s \subset \mathcal{F}_t\) whenever \(s \leq t\). For brevity, we denote \(\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\).

We interpret the \(\sigma\)-field \(\mathcal{F}_t\) as the information available to an agent at time \(t\). In particular, \(\mathcal{F}_0\) represents the information available at time 0, that is, the initial information.

We assume that the information accumulated over time can only grow, so that we never forget anything!
Definition (Stochastic Process)

A family \( X = (X_t)_{0 \leq t \leq T} \) of random variables is called a stochastic process. A stochastic process \( X \) is said to be \( \mathbb{F} \)-adapted if for every \( t = 0, 1, \ldots, T \) the random variable \( X_t \) is \( \mathcal{F}_t \)-measurable.

Example (5.3)

Consider once again the elementary market model. The stochastic process of the stock price is \( S_0 \) and \( S_1 \) on \( \Omega = \{\omega_1, \omega_2\} \) and the filtration is \( \mathcal{F}_0 = \{\emptyset, \Omega\} \) and

\[
\mathcal{F}_1 = \{\emptyset, \Omega, \{\omega_1\}, \{\omega_2\}\} = 2^\Omega.
\]

Note that \( \mathcal{F}_0 \) is the initial information, which means that the investor knows only all possible states.
Filtration Generated by a Stochastic Process

- The initial information at time 0 is usually given by the trivial $\sigma$-field $F_0 = \{\emptyset, \Omega\}$ since all prices are known at 0, so that there is no uncertainty.

- Let $X = (X_t)_{0 \leq t \leq T}$ be a stochastic process on the probability space $(\Omega, F, P)$.

- For a fixed $t$, we define the $\sigma$-field $F^X_t$ by setting

$$F^X_t = \sigma\left(X^{-1}_s([a, b]) \mid 0 \leq s \leq t, a \leq b\right).$$

Then the filtration $\mathbb{F}^X = (F^X_t)_{0 \leq t \leq T}$ is called the filtration generated by the process $X$. 
Example 5.4: Stock Price Model

\[ S_2 = 1.8 \quad \omega_1 \quad \mathbb{P}(\omega_1) = \frac{3}{16} \]

\[ S_1 = 1.3 \]

\[ S_2 = 1.2 \quad \omega_2 \quad \mathbb{P}(\omega_2) = \frac{5}{16} \]

\[ S_0 = 1 \]

\[ S_2 = 1.1 \quad \omega_3 \quad \mathbb{P}(\omega_3) = \frac{4}{16} \]

\[ S_1 = 0.7 \]

\[ S_2 = 0.5 \quad \omega_4 \quad \mathbb{P}(\omega_4) = \frac{4}{16} \]
Example (5.4 Continued)

- $\mathcal{F}_0^S = \{\emptyset, \Omega\}$ since $S_0$ is deterministic.
- At $t = 1$, we have

\[
S_1^{-1}([a, b]) = \begin{cases} 
\{\omega_1, \omega_2\} & \text{if } 0.7 < a \leq 1.3 \text{ and } b \geq 1.3 \\
\{\omega_3, \omega_4\} & \text{if } a \leq 0.7 \text{ and } 0.7 \leq b < 1.3 \\
\emptyset & \text{otherwise}
\end{cases}
\]

and thus

\[
\mathcal{F}_1^S = \left\{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\right\}.
\]

- $\mathcal{F}_2^S = 2^\Omega$ since the partition generating $\mathcal{F}_2$ is

\[
\mathcal{P}_2 = \left\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\right\}.
\]
PART 3

CONDITIONAL EXPECTATION
Conditional Expectation

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite (or countable) probability space.
- Let $X$ be an arbitrary $\mathcal{F}$-measurable random variable.
- Assume that $\mathcal{G}$ is a $\sigma$-field which is contained in $\mathcal{F}$.
- Let $(A_i)_{i \in I}$ be the unique partition associated with $\mathcal{G}$.
- Our next goal is to define the **conditional expectation** $\mathbb{E}_\mathbb{P}(X|\mathcal{G})$, that is, the conditional expectation of a random variable $X$ with respect to a $\sigma$-field $\mathcal{G}$.
- The expected value $\mathbb{E}_\mathbb{P}(X)$ will be obtained from $\mathbb{E}_\mathbb{P}(X|\mathcal{G})$ by setting $\mathcal{G} = \mathcal{F}_0$, that is, $\mathbb{E}_\mathbb{P}(X) = \mathbb{E}_\mathbb{P}(X|\mathcal{F}_0)$. 

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The **conditional expectation** $\mathbb{E}_\mathcal{P}(X \mid \mathcal{G})$ of $X$ with respect to $\mathcal{G}$ is defined as the random variable which satisfies, for every $\omega \in A_i$,

$$\mathbb{E}_\mathcal{P}(X \mid \mathcal{G})(\omega) = \frac{1}{\mathbb{P}(A_i)} \sum_{\omega_l \in A_i} X(\omega_l)\mathbb{P}(\omega_l) = \sum_{x_k} x_k \mathbb{P}(X = x_k \mid A_i)$$

where the summation is over all possible values of $X$ and

$$\mathbb{P}(X = x_k \mid A_i) = (\mathbb{P}(A_i))^{-1}\mathbb{P}(\{X = x_k\} \cap A_i)$$

is the conditional probability of the event $\{\omega \in \Omega \mid X(\omega) = x_k\}$ given $A_i$. Hence

$$\mathbb{E}_\mathcal{P}(X \mid \mathcal{G}) = \sum_{i \in I} \frac{1}{\mathbb{P}(A_i)} \mathbb{E}_\mathcal{P}(X 1_{A_i}) 1_{A_i}.$$
Properties of Conditional Expectation

- $\mathbb{E}_P(X|\mathcal{G})$ is well defined by equation and, by Proposition 5.1, the conditional expectation $\mathbb{E}_P(X|\mathcal{G})$ is a $\mathcal{G}$-measurable r.v.
- $\mathbb{E}_P(X|\mathcal{G})$ is the best estimate of $X$ given the information represented by the $\sigma$-field $\mathcal{G}$.
- The following identity uniquely characterises the conditional expectation (in addition to $\mathcal{G}$-measurability):

  \[
  \sum_{\omega \in \mathcal{G}} X(\omega) P(\omega) = \sum_{\omega \in \mathcal{G}} \mathbb{E}_P(X|\mathcal{G})(\omega) P(\omega), \quad \forall \mathcal{G} \in \mathcal{G}.
  \]

- One can represent this equality using (discrete) integrals: for every $\mathcal{G} \in \mathcal{G}$,

  \[
  \int_G X \, dP = \int_G \mathbb{E}_P(X|\mathcal{G}) \, dP, \quad \forall \mathcal{G} \in \mathcal{G}.
  \]
Proposition (5.2)

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be endowed with sub-\(\sigma\)-fields \(\mathcal{G}\) and \(\mathcal{G}_1 \subset \mathcal{G}_2\) of \(\mathcal{F}\). Then

1. **Tower property:** If \(X : \Omega \rightarrow \mathbb{R}\) is an \(\mathcal{F}\)-measurable r.v. then

\[
\mathbb{E}_\mathbb{P}(X | \mathcal{G}_1) = \mathbb{E}_\mathbb{P}(\mathbb{E}_\mathbb{P}(X | \mathcal{G}_2) | \mathcal{G}_1) = \mathbb{E}_\mathbb{P}(\mathbb{E}_\mathbb{P}(X | \mathcal{G}_1) | \mathcal{G}_2).
\]

2. **Taking out what is known:** If \(X : \Omega \rightarrow \mathbb{R}\) is a \(\mathcal{G}\)-measurable r.v. and \(Y : \Omega \rightarrow \mathbb{R}\) is an \(\mathcal{F}\)-measurable r.v. then

\[
\mathbb{E}_\mathbb{P}(XY | \mathcal{G}) = X \mathbb{E}_\mathbb{P}(Y | \mathcal{G}).
\]

3. **Trivial conditioning:** If \(X : \Omega \rightarrow \mathbb{R}\) is an \(\mathcal{F}\)-measurable r.v. independent of \(\mathcal{G}\) then

\[
\mathbb{E}_\mathbb{P}(X | \mathcal{G}) = \mathbb{E}_\mathbb{P}(X).
\]
Example 5.5: Conditional Expectation

\begin{align*}
S_2 &= 9 \\
\omega_1 &\quad \mathbb{P}(\omega_1) = 6/25 \\
S_1 &= 8 \\
S_2 &= 6 \\
\omega_2 &\quad \mathbb{P}(\omega_2) = 4/25 \\
S_0 &= 5 \\
S_2 &= 6 \\
\omega_3 &\quad \mathbb{P}(\omega_3) = 6/25 \\
S_1 &= 4 \\
S_2 &= 6 \\
\omega_4 &\quad \mathbb{P}(\omega_4) = 9/25
\end{align*}
Example (5.5)

The underlying probability space is given by $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

- At time $t = 0$, the stock price is known and unique value is $S_0 = 5$. Hence the $\sigma$-field $\mathcal{F}^S_0$ is the trivial $\sigma$-field.
- At time $t = 1$, the stock can take two possible values and

$$S_1^{-1}([a, b]) = \begin{cases} 
\Omega & \text{if } a \leq 4 \text{ and } 8 \leq b \\
\{\omega_1, \omega_2\} & \text{if } 4 < a \text{ and } 8 \leq b \\
\{\omega_3, \omega_4\} & \text{if } a \leq 4 \text{ and } b < 8 \\
\emptyset & \text{if } a < 4 \text{ and } b < 4 
\end{cases}$$

so that $\mathcal{F}^S_1 = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\}$.
- At time $t = 2$, we have $\mathcal{F}^S_1 = 2\Omega$. 
Example (5.5 Continued)

\[
\begin{align*}
\mathbb{P}(S_2 = 9 \mid A_1) &= \frac{\mathbb{P}\left(\{\omega \in A_1\} \cap \{S_2(\omega) = 9\}\right)}{\mathbb{P}(A_1)} = \frac{\mathbb{P}(\omega_1)}{\mathbb{P}(\{\omega_1, \omega_2\})} = \frac{6}{25} = \frac{3}{5} \\
\mathbb{P}(S_2 = 6 \mid A_1) &= \frac{\mathbb{P}\left(\{\omega \in A_1\} \cap \{S_2(\omega) = 6\}\right)}{\mathbb{P}(A_1)} = \frac{\mathbb{P}(\omega_2)}{\mathbb{P}(\{\omega_1, \omega_2\})} = \frac{4}{25} = \frac{2}{5} \\
\mathbb{P}(S_2 = 3 \mid A_1) &= \frac{\mathbb{P}\left(\{\omega \in A_1\} \cap \{S_2(\omega) = 3\}\right)}{\mathbb{P}(A_1)} = \frac{\mathbb{P}(\emptyset)}{\mathbb{P}(\{\omega_1, \omega_2\})} = 0 \\
\mathbb{P}(S_2 = 9 \mid A_2) &= \frac{\mathbb{P}\left(\{\omega \in A_2\} \cap \{S_2(\omega) = 9\}\right)}{\mathbb{P}(A_2)} = \frac{\mathbb{P}(\emptyset)}{\mathbb{P}(\{\omega_3, \omega_4\})} = 0 \\
\mathbb{P}(S_2 = 6 \mid A_2) &= \frac{\mathbb{P}\left(\{\omega \in A_2\} \cap \{S_2(\omega) = 6\}\right)}{\mathbb{P}(A_2)} = \frac{\mathbb{P}(\omega_3)}{\mathbb{P}(\{\omega_3, \omega_4\})} = \frac{6}{25} = \frac{3}{5} \\
\mathbb{P}(S_2 = 3 \mid A_2) &= \frac{\mathbb{P}\left(\{\omega \in A_2\} \cap \{S_2(\omega) = 3\}\right)}{\mathbb{P}(A_2)} = \frac{\mathbb{P}(\omega_4)}{\mathbb{P}(\{\omega_3, \omega_4\})} = \frac{9}{25} = \frac{3}{5}
\end{align*}
\]
Example: Conditional Expectation

Example (5.5 Continued)

We have

\[ \mathbb{E}_P(S_2 \mid \mathcal{F}_1^S)(\omega) = 9 \cdot \frac{3}{5} + 6 \cdot \frac{2}{5} + 3 \cdot 0 = \frac{39}{5} \quad \text{for } \omega \in A_1 \]

\[ \mathbb{E}_P(S_2 \mid \mathcal{F}_1^S)(\omega) = 9 \cdot 0 + 6 \cdot \frac{2}{5} + 3 \cdot \frac{3}{5} = \frac{21}{5} \quad \text{for } \omega \in A_2 \]

and thus

\[ \mathbb{E}_P(S_2 \mid \mathcal{F}_1^S) = \begin{cases} 
\frac{39}{5} & \text{if } \omega \in \{\omega_1, \omega_2\} \\
\frac{21}{5} & \text{if } \omega \in \{\omega_3, \omega_4\} 
\end{cases} \]

Note that

\[ \mathbb{E}_P\left(\mathbb{E}_P(S_2 \mid \mathcal{F}_1^S)\right) = \frac{39}{5} \cdot \frac{2}{5} + \frac{21}{5} \cdot \frac{3}{5} = \frac{141}{25} \]

\[ \mathbb{E}_P(S_2) = 9 \cdot \frac{6}{25} + 6 \cdot \frac{4}{25} + 6 \cdot \frac{6}{25} + 3 \cdot \frac{9}{25} = \frac{141}{25} \]
PART 4

CHANGE OF A PROBABILITY MEASURE
Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent probability measures on $(\Omega, \mathcal{F})$. Let the Radon-Nikodym density of $\mathbb{Q}$ with respect to $\mathbb{P}$ be

$$
\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) = L(\omega), \quad \mathbb{P}\text{-a.s.}
$$

meaning that $L$ is $\mathcal{F}$-measurable and, for every $A \in \mathcal{F}$,

$$
\int_A X \, d\mathbb{Q} = \int_A XL \, d\mathbb{P}.
$$

If $\Omega$ is finite then this equality becomes

$$
\sum_{\omega \in A} X(\omega) \mathbb{Q}(\omega) = \sum_{\omega \in A} X(\omega)L(\omega) \mathbb{P}(\omega).
$$

The r.v. $L$ is strictly positive $\mathbb{P}$-a.s. and $\mathbb{E}_\mathbb{P}(L) = 1$.

Equality $\mathbb{E}_\mathbb{Q}(X) = \mathbb{E}_\mathbb{P}(XL)$ holds for any $\mathbb{Q}$-integrable random variable $X$ (it suffices to take $A = \Omega$).
Lemma (5.1: Bayes Formula)

Let $\mathcal{G}$ be a sub-$\sigma$-field of $\mathcal{F}$ and let $X$ be a $\mathbb{Q}$-integrable random variable. Then the Bayes formula holds

$$
\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{G}) = \frac{\mathbb{E}_{\mathbb{P}}(XL \mid \mathcal{G})}{\mathbb{E}_{\mathbb{P}}(L \mid \mathcal{G})}.
$$

Proof.

- $\mathbb{E}_{\mathbb{P}}(L \mid \mathcal{G})$ is strictly positive so that the RHS is well defined.
- By our assumption, the random variable $XL$ is $\mathbb{P}$-integrable.
- Therefore, it suffices to show that the equality

$$
\mathbb{E}_{\mathbb{P}}(XL \mid \mathcal{G}) = \mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{G}) \mathbb{E}_{\mathbb{P}}(L \mid \mathcal{G})
$$

holds for every random variable $X$. 
Proof of Lemma 5.1 for General $\sigma$-fields (MATH3975)

Proof.

Since the RHS of the last formula defines a $G$-measurable random variable, it suffices to verify that for all sets $G \in \mathcal{G}$, we have

$$\int_G XL \, dP = \int_G E_Q(X | G)E_P(L | G) \, dP.$$

For every $G \in \mathcal{G}$, we obtain

$$\int_G XL \, dP \overset{L-density}{=} \int_G X \, dQ \overset{Q-cond.}{=} \int_G E_Q(X | G) \, dQ$$

$$\overset{L-density}{=} \int_G E_Q(X | G)L \, dP \overset{P-cond.}{=} \int_G E_P(E_Q(X | G)L | G) \, dP$$

$$\overset{Prop.5.2}{=} \int_G E_Q(X | G)E_P(L | G) \, dP.$$

This ends the proof of Lemma 5.1 for the general case.
Proof of Lemma 5.1 for Partitions (MATH3075)

Proof.

Now assume that $\mathcal{G} = \sigma(A_1, \ldots, A_m)$ and $Q(\omega_l) = L(\omega_l)\mathbb{P}(\omega_l)$. Then on every event $A_i$

$$
\mathbb{E}_Q(X|\mathcal{G}) = \frac{1}{Q(A_i)} \sum_{\omega_l \in A_i} X(\omega_l)Q(\omega_l)
$$

$$
\mathbb{E}_\mathbb{P}(LX|\mathcal{G}) = \frac{1}{\mathbb{P}(A_i)} \sum_{\omega_l \in A_i} L(\omega_l)X(\omega_l)\mathbb{P}(\omega_l) = \frac{1}{\mathbb{P}(A_i)} \sum_{\omega_l \in A_i} X(\omega_l)Q(\omega_l)
$$

$$
\mathbb{E}_\mathbb{P}(L|\mathcal{G}) = \frac{1}{\mathbb{P}(A_i)} \sum_{\omega_l \in A_i} L(\omega_l)\mathbb{P}(\omega_l) = \frac{Q(A_i)}{\mathbb{P}(A_i)}.
$$

Hence on every event $A_i$ from the partition $\mathcal{P} = \{A_1, \ldots, A_m\}$

$$
\mathbb{E}_Q(X|\mathcal{G}) = \frac{\mathbb{E}_\mathbb{P}(XL|\mathcal{G})}{\mathbb{E}_\mathbb{P}(L|\mathcal{G})}.
$$

This ends the proof of Lemma 5.1 when $\mathcal{G} = \sigma(A_1, \ldots, A_m)$. 