7: The CRR Market Model

Marek Rutkowski
School of Mathematics and Statistics
University of Sydney

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We will examine the following issues:

1. The Cox-Ross-Rubinstein Market Model
2. The CRR Call Option Pricing Formula
3. Call and Put Options of American Style
4. Dynamic Programming Approach to American Claims
5. Examples: American Call and Put Options
PART 1

THE COX-ROSS-RUBINSTEIN MARKET MODEL
The Cox-Ross-Rubinstein market model (CRR model) is an example of a multi-period market model of the stock price. At each point in time, the stock price is assumed to either go ‘up’ by a fixed factor $u$ or go ‘down’ by a fixed factor $d$.

Only three parameters are needed to specify the binomial asset pricing model: $u > d > 0$ and $r > -1$.

Note that we do not postulate that $d < 1 < u$.

The real-world probability of an ‘up’ movement is assumed to be the same $0 < p < 1$ for each period and is assumed to be independent of all previous stock price movements.
Bernoulli Processes

**Definition (Bernoulli Process)**

A process \( X = (X_t)_{1 \leq t \leq T} \) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is called the **Bernoulli process** with parameter \(0 < p < 1\) if the random variables \(X_1, X_2, \ldots, X_T\) are independent and have the following common probability distribution

\[
\mathbb{P}(X_t = 1) = 1 - \mathbb{P}(X_t = 0) = p.
\]

**Definition (Bernoulli Counting Process)**

The **Bernoulli counting process** \(N = (N_t)_{0 \leq t \leq T}\) is defined by setting \(N_0 = 0\) and, for every \(t = 1, \ldots, T\) and \(\omega \in \Omega\),

\[
N_t(\omega) := X_1(\omega) + X_2(\omega) + \cdots + X_t(\omega).
\]

The process \(N\) is a special case of an **additive random walk**.
The stock price process in the CRR model is defined via an initial value $S_0 > 0$ and, for $1 \leq t \leq T$ and all $\omega \in \Omega$,

$$S_t(\omega) := S_0 u^{N_t(\omega)} d^{t-N_t(\omega)}.$$

- The underlying Bernoulli process $X$ governs the ‘up’ and ‘down’ movements of the stock. The stock price moves up at time $t$ if $X_t(\omega) = 1$ and moves down if $X_t(\omega) = 0$.
- The dynamics of the stock price can be seen as an example of a **multiplicative random walk**.
- The Bernoulli counting process $N$ counts the up movements. Before and including time $t$, the stock price moves up $N_t$ times and down $t - N_t$ times.
Distribution of the Stock Price

- For each \( t = 1, 2, \ldots, T \), the random variable \( N_t \) has the \textbf{binomial distribution} with parameters \( p \) and \( t \).

- Specifically, for every \( t = 1, \ldots, T \) and \( k = 0, \ldots, t \) we have that
  \[
  \mathbb{P}(N_t = k) = \binom{t}{k} p^k (1 - p)^{t-k}.
  \]

- Hence the probability distribution of the stock price \( S_t \) at time \( t \) is given by
  \[
  \mathbb{P}(S_t = S_0 u^k d^{t-k}) = \binom{t}{k} p^k (1 - p)^{t-k}
  \]
  for \( k = 0, 1, \ldots, t \).
Figure: Stock Price Lattice in the CRR Model
Proposition (7.1)

Assume that $d < 1 + r < u$. Then a probability measure $\tilde{P}$ on $(\Omega, \mathcal{F}_T)$ is a risk-neutral probability measure for the CRR model $\mathcal{M} = (B, S)$ with parameters $p, u, d, r$ and time horizon $T$ if and only if:

1. $X_1, X_2, X_3, \ldots, X_T$ are independent under the probability measure $\tilde{P}$,
2. $0 < \tilde{p} := \tilde{P}(X_t = 1) < 1$ for all $t = 1, \ldots, T$,
3. $\tilde{p}u + (1 - \tilde{p})d = (1 + r),$

where $X$ is the Bernoulli process governing the stock price $S$. 
Proposition (7.2)

If \( d < 1 + r < u \) then the CRR market model \( \mathcal{M} = (B, S) \) is arbitrage-free and complete.

- Since the CRR model is complete, the unique arbitrage price of any European contingent claim can be computed using the risk-neutral valuation formula

\[
\pi_t(X) = B_t \mathbb{E}_{\tilde{P}} \left( \frac{X}{B_T} \middle| \mathcal{F}_t \right)
\]

- We will apply this formula to the call option on the stock.
PART 2

THE CRR CALL OPTION PRICING FORMULA
Proposition (7.3)

The arbitrage price at time $t = 0$ of the European call option $C_T = (S_T - K)^+$ in the binomial market model $\mathcal{M} = (B, S)$ is given by the CRR call pricing formula

$$C_0 = S_0 \sum_{k=\hat{k}}^{T} \binom{T}{k} \hat{p}^k (1 - \hat{p})^{T-k} - \frac{K}{(1 + r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \tilde{p}^k (1 - \tilde{p})^{T-k}$$

where

$$\tilde{p} = \frac{1 + r - d}{u - d}, \quad \hat{p} = \frac{\tilde{p}u}{1 + r}$$

and $\hat{k}$ is the smallest integer $k$ such that

$$k \log \left( \frac{u}{d} \right) > \log \left( \frac{K}{S_0 d^T} \right).$$
Proof of Proposition 7.3.

- The price at time $t = 0$ of the claim $X = C_T = (S_T - K)^+$ can be computed using the risk-neutral valuation under $\tilde{P}$

\[ C_0 = \frac{1}{(1 + r)^T} \mathbb{E}_{\tilde{P}}(C_T). \]

- In view of Proposition 7.1, we obtain

\[ C_0 = \frac{1}{(1 + r)^T} \sum_{k=0}^{T} \binom{T}{k} \tilde{p}^k (1 - \tilde{p})^{T-k} \max(0, S_0 u^k d^{T-k} - K). \]

- We note that

\[ S_0 u^k d^{T-k} - K > 0 \iff \left( \frac{u}{d} \right)^k > \frac{K}{S_0 d^T} \]

\[ \iff k \log \left( \frac{u}{d} \right) > \log \left( \frac{K}{S_0 d^T} \right) \]
Proof of Proposition 7.3 (Continued).

- We define $\hat{k} = \hat{k}(S_0, T)$ as the smallest integer $k$ such that the last inequality is satisfied. If there are less than $\hat{k}$ upward mouvements there is no chance that the option will expire worthless.

- Therefore, we obtain

$$C_0 = \frac{1}{(1 + r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \tilde{p}^k (1 - \tilde{p})^{T-k} \left( S_0 u^k d^{T-k} - K \right)$$

$$= \frac{S_0}{(1 + r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \tilde{p}^k (1 - \tilde{p})^{T-k} u^k d^{T-k}$$

$$- \frac{K}{(1 + r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \tilde{p}^k (1 - \tilde{p})^{T-k}$$
Consequently,

\[ C_0 = S_0 \sum_{k=\hat{k}}^{T} \binom{T}{k} \left( \frac{\tilde{p}u}{1+r} \right)^k \left( \frac{(1-\tilde{p})d}{1+r} \right)^{T-k} \]

\[ - \frac{K}{(1+r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \tilde{p}^k (1-\tilde{p})^{T-k} \]

and thus

\[ C_0 = S_0 \sum_{k=\hat{k}}^{T} \binom{T}{k} \hat{p}^k (1-\hat{p})^{T-k} - \frac{K}{(1+r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \tilde{p}^k (1-\tilde{p})^{T-k} \]

where we denote \( \hat{p} = \frac{\tilde{p}u}{1+r} \).
Check that $0 < \hat{p} = \frac{\tilde{p}u}{1+r} < 1$ whenever $0 < \tilde{p} = \frac{1+r-d}{u-d} < 1$.

Let $\hat{P}$ be the probability measure obtained by setting $p = \hat{p}$ in Proposition 7.1. Then the process $\frac{B}{S}$ is a martingale under $\hat{P}$.

For $t = 0$, the price of the call satisfies

$$C_0 = S_0 \hat{P}(D) - KB(0, T) \tilde{P}(D)$$

where $D = \{\omega \in \Omega : S_T(\omega) > K\}$.

Recall that

$$C_t = B_t \mathbb{E}_{\tilde{P}}(B_T^{-1}(S_T - K)^+ | \mathcal{F}_t)$$

Using the abstract Bayes formula, one can show that

$$C_t = S_t \hat{P}(D | \mathcal{F}_t) - KB(t, T) \tilde{P}(D | \mathcal{F}_t).$$
It is possible to derive explicit pricing formula for the call option at any date \( t = 0, 1, \ldots, T \).

Since \( C_T - P_T = S_T - K \), we see that the following put-call parity holds at any date \( t = 0, 1, \ldots, T \)

\[
C_t - P_t = S_t - K(1 + r)^{-(T-t)} = S_t - KB(t, T)
\]

where

\[
B(t, T) = (1 + r)^{-(T-t)}
\]

is the price at time \( t \) of zero-coupon bond maturing at \( T \).

Using Proposition 7.3 and the put-call parity, one can derive an explicit pricing formula for the European put option with the payoff \( P_T = (K - S_T)^+ \). This is left as an exercise.
PART 3

CALL AND Put OPTIONS OF AMERICAN STYLE
In contrast to a contingent claim of a European style, a claim of an American style can be exercised by its holder at any date before its expiration date $T$.

**Definition (American Call and Put Options)**

An American call (put) option is a contract which gives the holder the right to buy (sell) an asset at any time $t \leq T$ at strike price $K$.

- It the study of an American claim, we are concerned with the price process and the ‘optimal’ exercise policy by its holder.
- If the holder of an American option exercises it at $\tau \in [0, T]$, $\tau$ is called the **exercise time**.
An admissible exercise time should belong to the class of stopping times.

**Definition (Stopping Time)**

A **stopping time** with respect to a given filtration $\mathcal{F}$ is a map $\tau : \Omega \rightarrow \{0, 1, \ldots, T\}$ such that for any $t = 0, 1, \ldots, T$ the event $\{\omega \in \Omega | \tau(\omega) = t\}$ belongs to the $\sigma$-field $\mathcal{F}_t$.

Intuitively, this property means that the decision whether to stop a given process at time $t$ (for instance, whether to exercise an option at time $t$ or not) depends on the stock price fluctuations up to time $t$ only.

**Definition**

Let $\mathcal{T}_{[t, T]}$ be the subclass of stopping times $\tau$ with respect to $\mathcal{F}$ satisfying the inequalities $t \leq \tau \leq T$. 

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American Call Option

**Definition**

By an **arbitrage price** of the American call we mean a price process $C_t^a$, $t \leq T$, such that the extended financial market model – that is, a market with trading in riskless bonds, stocks and an American call option – remains arbitrage-free.

**Proposition (7.4)**

The price of an American call option in the CRR arbitrage-free market model with $r \geq 0$ coincides with the arbitrage price of a European call option with the same expiry date and strike price.

**Proof.**

It is sufficient to show that the American call option should never be exercised before maturity, since otherwise the issuer of the option would be able to make riskless profit.
Proof of Proposition 7.4

Proof of Proposition 7.4 – Step 1.

- The argument hinges on the inequality, for $t = 0, 1, \ldots, T$,

\[ C_t \geq (S_t - K)^+. \]  \hspace{1cm} (1)

- An intuitive way of deriving (1) is based on no-arbitrage arguments.

- Notice that since the price $C_t$ is always non-negative, it suffices to consider the case when the current stock price is greater than the exercise price: $S_t - K > 0$.

- Suppose, on the contrary, that $C_t < S_t - K$ for some $t$.

- Then it would be possible, with zero net initial investment, to buy at time $t$ a call option, short a stock, and invest the sum $S_t - C_t > K$ in the savings account.
Proof of Proposition 7.4 – Step 1.

- By holding this portfolio unchanged up to the maturity date $T$, we would be able to lock in a riskless profit.

- Indeed, the value of our portfolio at time $T$ would satisfy (recall that $r \geq 0$)

\[
C_T - S_T + (1 + r)^{T-t}(S_t - C_t) > (S_T - K)^+ - S_T + (1 + r)^{T-t}K \geq 0.
\]

- We conclude that inequality (1) is necessary for the absence of arbitrage opportunities.

- In the next step, we assume that (1) holds.
Proof of Proposition 7.4 – Step 2.

- Taking (1) for granted, we may now deduce the property $C_t^a = C_t$ using simple no-arbitrage arguments.

- Suppose, on the contrary, that the issuer of an American call is able to sell the option at time 0 at the price $C_0^a > C_0$.

- In order to profit from this transaction, the option writer establishes a dynamic portfolio $\phi$ that replicates the value process of the European call and invests the remaining funds in the savings account.

- Suppose that the holder of the option decides to exercise it at instant $t$ before the expiry date $T$. 
Proof of Proposition 7.4 – Step 2.

Then the issuer of the option locks in a riskless profit, since the value of his portfolio at time $t$ satisfies

$$C_t - (S_t - K)^+ + (1 + r)^t(C_0^a - C_0) > 0.$$ 

The above reasoning implies that the European and American call options are equivalent from the point of view of arbitrage pricing theory.

Both options have the same price and an American call should never be exercised by its holder before expiry.

Note that the assumption $r \geq 0$ was necessary to obtain (1).
American Put Option

- Recall that the American put is an option to sell a specified number of shares, which may be exercised at any time before or at the expiry date $T$.
- For the American put on stock with strike $K$ and expiry date $T$, we have the following valuation result.

**Proposition (7.5)**

The arbitrage price $P^a_t$ of an American put option equals

$$P^a_t = \max_{\tau \in [t, T]} \mathbb{E}_{\tilde{\mathbb{P}}} ((1 + r)^{-(\tau - t)} (K - S_\tau)^+) \mid \mathcal{F}_t), \quad \forall \ t \leq T.$$  

For any $t \leq T$, the stopping time $\tau^*_t$ which realizes the maximum is given by the expression

$$\tau^*_t = \min \{ u \geq t \mid P^a_u = (K - S_u)^+ \}.$$
PART 4

DYNAMIC PROGRAMMING APPROACH
TO AMERICAN CLAIMS
The stopping time $\tau_t^*$ is called the rational exercise time of an American put option that is assumed to be still alive at time $t$.

By an application of the classic Bellman principle (1952), one reduces the optimal stopping problem in Proposition 7.5 to an explicit recursive procedure for the value process.

The following corollary to Proposition 7.5 gives the dynamic programming recursion for the value of an American put option.

Note that this is an extension of the backward induction approach to the valuation of European contingent claims.
Corollary (Bellman Principle)

Let the non-negative adapted process $U$ be defined recursively by setting $U_T = (K - S_T)^+$ and for $t \leq T - 1$

$$U_t = \max \left\{ (K - S_t)^+, (1 + r)^{-1} \mathbb{E}^{\tilde{P}}_{F_t}(U_{t+1} \mid F_t) \right\}.$$ 

Then the arbitrage price $P^a_t$ of the American put option at time $t$ equals $U_t$ and the rational exercise time after time $t$ admits the following representation

$$\tau^*_t = \min \left\{ u \geq t \mid U_u = (K - S_u)^+ \right\}.$$ 

Therefore, $\tau^*_T = T$ and for every $t = 0, 1, \ldots, T - 1$

$$\tau^*_t = t \mathbb{1}_{\{U_t=(K-S_t)^+\}} + \tau^*_t + 1 \mathbb{1}_{\{U_t>(K-S_t)^+\}}.$$
Dynamic Programming Recursion

- It is also possible to show directly that the price $P_t^a$ satisfies the recursive relationship, for $t \leq T - 1$,

$$P_t^a = \max \left\{ (K - S_t)^+, (1 + r)^{-1} \mathbb{E}_P^\sim (P_{t+1}^a | \mathcal{F}_t) \right\}$$

subject to the terminal condition $P_T^a = (K - S_T)^+$.

- In the case of the CRR model, this formula reduces the valuation problem to the simple single-period case.

- To show this we shall argue by contradiction. Assume first that (2) fails to hold for $t = T - 1$. If this is the case, one may easily construct at time $T - 1$ a portfolio which produces riskless profit at time $T$. Hence, we conclude that necessarily

$$P_{T-1}^a = \max \left\{ (K - S_{T-1})^+, (1 + r)^{-1} \mathbb{E}_P^\sim ((K - S_T)^+ | \mathcal{F}_T) \right\}.$$ 

- This procedure may be repeated as many times as needed.

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American Put Option: Summary

To summarise:

- In the CRR model, the arbitrage pricing of the American put option reduces to the following recursive recipe, for $t \leq T - 1$,

$$P_t^a = \max \left\{ (K - S_t)^+, (1 + r)^{-1}(\tilde{p}P_{t+1}^{au} + (1 - \tilde{p})P_{t+1}^{ad}) \right\}$$

with the terminal condition

$$P_T^a = (K - S_T)^+.$$

- The quantities $P_{t+1}^{au}$ and $P_{t+1}^{ad}$ represent the values of the American put in the next step corresponding to the upward and downward movements of the stock price starting from a given node on the CRR lattice.
An **American contingent claim** $X^a = (X, \mathcal{T}_{[0, T]})$ expiring at $T$ consists of a sequence of payoffs $(X_t)_{0 \leq t \leq T}$ where the random variable $X_t$ is $\mathcal{F}_t$-measurable for $t = 0, 1, \ldots, T$ and the set $\mathcal{T}_{[0, T]}$ of admissible exercise policies.

- We interpret $X_t$ as the **payoff** received by the holder of the claim $X^a$ upon exercising it at time $t$.
- The set of admissible exercise policies is restricted to the class $\mathcal{T}_{[0, T]}$ of all **stopping times** with values in $\{0, 1, \ldots, T\}$.
- Let $g : \mathbb{R} \times \{0, 1, \ldots, T\} \rightarrow \mathbb{R}$ be an arbitrary function. We say that $X^a$ is a **path-independent** American claim with the payoff function $g$ if the equality $X_t = g(S_t, t)$ holds for every $t = 0, 1, \ldots, T$. 
Proposition (7.6)

For every $t \leq T$, the arbitrage price $\pi(X^a)$ of an American claim $X^a$ in the CRR model equals

$$\pi_t(X^a) = \max_{\tau \in T_{[t,T]}} \mathbb{E}_{\tilde{P}}((1 + r)^{-(\tau-t)}X_{\tau} | \mathcal{F}_t).$$

The price process $\pi(X^a)$ satisfies the following recurrence relation, for $t \leq T - 1$,

$$\pi_t(X^a) = \max \left\{ X_t, \mathbb{E}_{\tilde{P}}((1 + r)^{-1}\pi_{t+1}(X^a) | \mathcal{F}_t) \right\}$$

with $\pi_T(X^a) = X_T$ and the rational exercise time $\tau^*_t$ equals

$$\tau^*_t = \min \left\{ u \geq t \mid X_u \geq \mathbb{E}_{\tilde{P}}((1 + r)^{-1}\pi_{u+1}(X^a) | \mathcal{F}_u) \right\}.$$
For a generic value of the stock price $S_t$ at time $t$, we denote by $\pi_{t+1}^u(X^a)$ and $\pi_{t+1}^d(X^a)$ the values of the price $\pi_{t+1}(X^a)$ at the nodes corresponding to the upward and downward movements of the stock price during the period $[t, t + 1]$, that is, for the values $uS_t$ and $dS_t$ of the stock price at time $t + 1$, respectively.

**Proposition (7.7)**

For a path-independent American claim $X^a$ with the payoff process $X_t = g(S_t, t)$ we obtain, for every $t \leq T - 1$,

$$\pi_t(X^a) = \max \left\{ g(S_t, t), (1+r)^{-1}(\tilde{p} \pi_{t+1}^u(X^a) + (1-\tilde{p}) \pi_{t+1}^d(X^a)) \right\}.$$
Consider a path-independent American claim $X^a$ with the payoff function $g(S_t, t)$. Then:

- Let $X^a_t = \pi_t(X^a)$ be the arbitrage price at time $t$ of $X^a$.
- Then the pricing formula becomes

$$X^a_t = \max \left\{ g(S_t, t), (1 + r)^{-1}(\tilde{p}X^a_{t+1} + (1 - \tilde{p})X^a_{t+1}) \right\}$$

with the terminal condition $X^a_T = g(S_T, T)$. Moreover

$$\tau^*_t = \min \left\{ u \geq t \mid g(S_u, u) \geq X^a_u \right\}.$$ 

- The risk-neutral valuation formula given above is valid for an arbitrary path-independent American claim with a payoff function $g$ in the CRR binomial model.
We consider here the CRR binomial model with the horizon date $T = 2$ and the risk-free rate $r = 0.2$.

The stock price $S$ for $t = 0$ and $t = 1$ equals

$$S_0 = 10, \quad S_1^u = 13.2, \quad S_1^d = 10.8.$$ 

Let $X^a$ be the American call option with maturity date $T = 2$ and the following payoff process

$$g(S_t, t) = (S_t - K_t)^+.$$ 

The strike $K_t$ is variable and satisfies

$$K_0 = 9, \quad K_1 = 9.9, \quad K_2 = 12.$$
Example (7.1 Continued)

- We will first compute the arbitrage price $\pi_t(X^a)$ of this option at times $t = 0, 1, 2$ and the rational exercise time $\tau_{0}^*$. 
- Subsequently, we will compute the replicating strategy for $X^a$ up to the rational exercise time $\tau_{0}^*$. 
- We start by noting that the unique risk-neutral probability measure $\tilde{P}$ satisfies

$$\tilde{p} = \frac{1 + r - d}{u - d} = \frac{(1 + r)S_0 - S_1^d}{S_1^u - S_1^d} = \frac{12 - 10.8}{13.2 - 10.8} = 0.5$$

- The dynamics of the stock price under $\tilde{P}$ are given by the first exhibit (note that $S_2^{ud} = S_2^{du}$). 
- The second exhibit represents the price of the call option.
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7: The CRR Market Model
Example (7.1 Continued)

**Holder.** The rational holder should exercise the American option at time $t = 1$ if the stock price rises during the first period. Otherwise, the option should be held till time 2. Hence $\tau^*_0 : \Omega \to \{0, 1, 2\}$ equals

\[
\begin{align*}
\tau^*_0(\omega) &= 1 \quad \text{for} \quad \omega \in \{\omega_1, \omega_2\} \\
\tau^*_0(\omega) &= 2 \quad \text{for} \quad \omega \in \{\omega_3, \omega_4\}
\end{align*}
\]

**Issuer.** We now take the position of the issuer of the option. At $t = 0$, we need to solve

\[
\begin{align*}
1.2 \phi^0_0 + 13.2 \phi^1_0 &= 3.3 \\
1.2 \phi^0_0 + 10.8 \phi^1_0 &= 0.94
\end{align*}
\]

Hence $(\phi^0_0, \phi^1_0) = (-8.067, 0.983)$ for all $\omega$. 
Example (7.1 Continued)

- If the stock price rises during the first period, the option is exercised and thus we do not need to compute the strategy at time 1 for \( \omega \in \{\omega_1, \omega_2\} \).
- If the stock price falls during the first period, we solve

\[
1.2 \tilde{\phi}^0_1 + 14.256 \phi^1_1 = 2.256 \\
1.2 \tilde{\phi}^0_1 + 11.664 \phi^1_1 = 0
\]

- Hence \((\tilde{\phi}^0_1, \phi^1_1) = (-8.46, 0.8704)\) for \( \omega \in \{\omega_3, \omega_4\} \).
- Note that \(\tilde{\phi}^0_1 = -8.46\) is the amount of cash borrowed at time 1, rather than the number of units of the savings account \(B\).
- The replicating strategy \(\phi = (\phi^0, \phi^1)\) is defined at time 0 for all \(\omega\) and it is defined at time 1 on the event \(\{\omega_3, \omega_4\}\) only.
\( V_1^\mu(\phi) = 3.3 \)

\( (\phi^0_0, \phi^0_0) = (-8.067, 0.983) \)

\( V_2^{du}(\phi) = 2.256 \)

\( (\phi^0_1, \phi^0_1) = (-8.46, 0.8704) \)

\( V_2^{dd}(\phi) = 0 \)

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PART 5

IMPLEMENTATION OF THE CRR MODEL
Derivation of \( u \) and \( d \) from \( r \) and \( \sigma \)

- We fix maturity \( T \) and we assume that the **continuously compounded** interest rate \( r \) is such that \( B(0, T) = e^{-rT} \).
- From the market data for stock prices, one can estimate the stock price **volatility** \( \sigma \) per one time unit (typically, one year).
- Note that until now we assumed that \( t = 0, 1, 2, \ldots, T \), which means that \( \Delta t = 1 \). In general, the length of each period can be any positive number smaller than 1. We set \( n = T/\Delta t \).
- Two widely used conventions for obtaining \( u \) and \( d \) from \( \sigma \) and \( r \) are:
  - **The Cox-Ross-Rubinstein (CRR) parametrisation**:
    
    \[
    u = e^{\sigma \sqrt{\Delta t}} \quad \text{and} \quad d = \frac{1}{u}.
    \]
  - **The Jarrow-Rudd (JR) parameterisation**:
    
    \[
    u = e^{(r - \frac{\sigma^2}{2})\Delta t + \sigma \sqrt{\Delta t}} \quad \text{and} \quad d = e^{(r - \frac{\sigma^2}{2})\Delta t - \sigma \sqrt{\Delta t}}.
    \]
Proposition (7.8)

Assume that $B_{k\Delta t} = (1 + r\Delta t)^k$ for every $k = 0, 1, \ldots, n$ and $u = \frac{1}{d} = e^{\sigma\sqrt{\Delta t}}$ in the CRR model. Then the risk-neutral probability measure $\tilde{P}$ satisfies

$$\tilde{P}(S_{t+\Delta t} = S_t u | S_t) = \frac{1}{2} + \frac{r - \frac{\sigma^2}{2}}{2\sigma}\sqrt{\Delta t} + o(\sqrt{\Delta t})$$

provided that $\Delta t$ is sufficiently small.

Proof of Proposition 7.8.

The risk-neutral probability measure for the CRR model is given by

$$\tilde{p} = \tilde{P}(S_{t+\Delta t} = S_t u | S_t) = \frac{1 + r\Delta t - d}{u - d}$$
Proof of Proposition 7.8.

Under the CRR parametrisation, we obtain

$$\tilde{p} = \frac{1 + r\Delta t - d}{u - d} = \frac{1 + r\Delta t - e^{-\sigma \sqrt{\Delta t}}}{e^{\sigma \sqrt{\Delta t}} - e^{-\sigma \sqrt{\Delta t}}}.$$

The Taylor expansions up to the second order term are

$$e^{\sigma \sqrt{\Delta t}} = 1 + \sigma \sqrt{\Delta t} + \frac{\sigma^2}{2} \Delta t + o(\Delta t)$$

$$e^{-\sigma \sqrt{\Delta t}} = 1 - \sigma \sqrt{\Delta t} + \frac{\sigma^2}{2} \Delta t + o(\Delta t)$$
Proof of Proposition 7.8.

By substituting the Taylor expansions into the risk-neutral probability measure, we obtain

\[
\tilde{p} = \frac{1 + r\Delta t - \left(1 - \sigma \sqrt{\Delta t} + \frac{\sigma^2}{2} \Delta t\right) + o(\Delta t)}{\left(1 + \sigma \sqrt{\Delta t} + \frac{\sigma^2}{2} \Delta t\right) - \left(1 - \sigma \sqrt{\Delta t} + \frac{\sigma^2}{2} \Delta t\right) + o(\Delta t)}
\]

\[
= \frac{\sigma \sqrt{\Delta t} + \left(r - \frac{\sigma^2}{2}\right) \Delta t + o(\Delta t)}{2\sigma \sqrt{\Delta t} + o(\Delta t)}
\]

\[
= \frac{1}{2} + \frac{r - \frac{\sigma^2}{2}}{2\sigma} \sqrt{\Delta t} + o(\sqrt{\Delta t})
\]

as was required to show.
To summarise, for $\Delta t$ sufficiently small, we get

$$\tilde{p} = \frac{1 + r\Delta t - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \approx \frac{1}{2} + \frac{r - \frac{\sigma^2}{2}}{2\sigma}\sqrt{\Delta t}.$$ 

Note that $1 + r\Delta t \approx e^{r\Delta t}$ when $\Delta t$ is sufficiently small.

Hence the risk-neutral probability measure can also be represented as follows

$$\tilde{p} \approx \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}.$$ 

More formally, if we define $\hat{r}$ such that $(1 + \hat{r})^n = e^{rT}$ for a fixed $T$ and $n = T/\Delta t$ then $\hat{r} \approx r\Delta t$ since $\ln(1 + \hat{r}) = r\Delta t$ and $\ln(1 + \hat{r}) \approx \hat{r}$ when $\hat{r}$ is close to zero.
Let the annualized variance of logarithmic returns be $\sigma^2 = 0.1$.
The interest rate is set to $r = 0.1$ per annum.
Suppose that the current stock price is $S_0 = 50$.
We examine European and American put options with strike price $K = 53$ and maturity $T = 4$ months (i.e. $T = \frac{1}{3}$).
The length of each period is $\Delta t = \frac{1}{12}$, that is, one month.
Hence $n = \frac{T}{\Delta t} = 4$ steps.
We adopt the CRR parameterisation to derive the stock price.
Then $u = 1.0956$ and $d = 1/u = 0.9128$.
We compute $1 + r\Delta t = 1.00833 \approx e^{r\Delta t}$ and $\tilde{p} = 0.5228$. 
Example: American Put Option

Example (7.2 Continued)

Figure: Stock Price Process

7: The CRR Market Model
Example: American Put Option

Example (7.2 Continued)

Figure: European Put Option Price

7: The CRR Market Model
Example: American Put Option

Example (7.2 Continued)

Figure: American Put Option Price
Example: American Put Option

Example (7.2 Continued)

Figure: Rational Exercise Policy
The next result deals with the Jarrow-Rudd parametrisation.

Proposition (7.9)

Let \( B_{k\Delta t} = (1 + r\Delta t)^k \) for \( k = 0, 1, \ldots, n \). We assume that

\[
    u = e^{(r - \frac{\sigma^2}{2})\Delta t + \sigma \sqrt{\Delta t}}
\]

and

\[
    d = e^{(r - \frac{\sigma^2}{2})\Delta t - \sigma \sqrt{\Delta t}}.
\]

Then the risk-neutral probability measure \( \tilde{P} \) satisfies

\[
    \tilde{P}(S_{t+\Delta t} = S_t u | S_t) = \frac{1}{2} + o(\Delta t)
\]

provided that \( \Delta t \) is sufficiently small.
Proof of Proposition 7.9.

Under the JR parameterisation, we have

\[
\tilde{p} = 1 + \frac{r \Delta t - d}{u - d} = 1 + r \Delta t - e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma \sqrt{\Delta t}} \frac{e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma \sqrt{\Delta t}} - e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma \sqrt{\Delta t}}}{e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma \sqrt{\Delta t}} - e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma \sqrt{\Delta t}}}.
\]

The Taylor expansions up to the second order term are

\[
e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma \sqrt{\Delta t}} = 1 + r \Delta t + \sigma \sqrt{\Delta t} + o(\Delta t)
\]

\[
e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma \sqrt{\Delta t}} = 1 + r \Delta t - \sigma \sqrt{\Delta t} + o(\Delta t)
\]

and thus

\[
\tilde{p} = \frac{1}{2} + o(\Delta t).
\]
Example (7.3 – JR Parameterisation)

- We consider the same problem as in Example 7.2, but with parameters $u$ and $d$ computed using the JR parameterisation. We obtain $u = 1.1002$ and $d = 0.9166$.
- As before, $1 + r\Delta t = 1.00833 \approx e^{r\Delta t}$, but $\tilde{p} = 0.5$.
- We compute the price processes for the stock, the European put option, the American put option and we find the rational exercise time.
- When we compare with Example 7.2, we see that the results are slightly different than before, although it appears that the rational exercise policy is the same.
- The CRR and JR parameterisations are both set to approach the Black-Scholes model.
- For $\Delta t$ sufficiently small, the prices computed under the two parametrisations will be very close to one another.
Example (7.3 Continued)

Figure: Stock Price Process
Example (7.3 Continued)

![Price Process of the European Put Option](image)

**Figure:** European Put Option Price

7: The CRR Market Model
Example: American Put Option

Example (7.3 Continued)

Price Process of the American Put Option

Figure: American Put Option Price
Example: American Put Option

Example (7.3 Continued)

1: Exercise the American Put; 0: Hold the American Option

Figure: Rational Exercise Policy

7: The CRR Market Model