THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Quiz 2

MATH3075/3975: Financial Mathematics (Normal) Semester 2, 2009

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3075/
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This take-home quiz is due at 5:00pm on Friday 23^{th} October

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1. Consider the portfolio

$$\mathcal{P} = \begin{cases} \text{long} & 3 & \text{stock share } (X) \\ \text{short} & 3 & \text{strike-}k_1 \text{ call } (C_1) \\ \text{short} & 1 & \text{strike-}k_2 \text{ call } (C_2) \end{cases},$$

where $k_1 < k_2$, and $\hat{\pi} = 3\hat{X}_0 - 3\hat{C}_1^0 - \hat{C}_2^0$ denotes face value to expiry of the market cost of this portfolio.

(a) Derive the expiry Profit-Loss-profile $Z(X)$ as a function of the share price X for this portfolio in terms of $\hat{\pi}$, k_1, k_2 ($k_2 > k_1$). Sketch Z as a function of X.

(b) Obtain the maximum profit, the maximum loss and the break even point(s).

(c) Determine any necessary condition(s) for this portfolio to be arbitrage-free.

(d) What is the market view of such an investor?

(e) Is this portfolio risky? Explain your answer.

2. Prove using arbitrage tables, the following inequality for the value of a European put option at time t with strike k, constant interest rate r and expiry time T

$$P(x,t;k) \leq ke^{-r(T-t)} ,$$

where x is the present value of the stock share.

3. Assume that a stock price follows a standard binomial process with parameters

$$x = \$10\,, \qquad u = 1.1\,, \qquad d = 0.9\,, \qquad r = 0\,.$$

A European derivative V with 2-periods to expiry written on the stock pays

$$V_T = \frac{1}{2}[X_T] + X_T\,\mathbb{I}(X_T \leq 9)\,.$$

Here $\mathbb{I}$ denotes the indicator function, and $[X]$ is a function which returns the nearest integer to X.

(a) Draw the stock and derivative trees to two time steps and determine the fair value V_0 of the derivative.

(b) If the derivative is currently trading for \$8 what should you do to take advantage of this mispricing? (You do not need to write down the exact numbers for the arbitrage strategy).

4. Consider the function

$$f(\xi) = \gamma \xi^2 \mathbb{I}(\xi < a) \,,$$

where $\mathbb{I}$ denotes the indicator function.

(a) Determine γ such that $f(\xi)$ is a valid probability density function on $\mathbb{R}^+$ for $a > 0$.

(b) Assume that $f(\xi)$ is the risk-neutral measure for the stock price with current value x, expiry T. Let r be the constant interest rate. Determine γ as a function of x, r and T using the risk-neutral restriction.

(c) Calculate the arbitrage-free present value of the following European derivative security with expiry T

$$V_T = X_T \, \mathbb{I}(X_T < \frac{1}{2}a) \,,$$

using the risk-free pdf determined in parts (a) and (b).

5. The price of a stock XYZ follows geometrical Brownian motion with constant volatility σ and riskless interest rate r. A derivative security written on this stock has payoff at expiry T given by

$$V_T = \frac{1}{X_T} \log(X_T/k) \,,$$

where X_T is the stock price at expiry and $k > 0$ is a constant.

Find the present value $V(x,t)$ of the derivative at any time $t < T$ given the stock price at time t is x. Express your answer in terms of $\tau := T - t$ (not t).