THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Quiz 2

MATH3075/3975: Financial Mathematics (Advanced) Semester 2, 2009

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3075/
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This take-home quiz is due at 5:00pm on Friday 23th October

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1. Consider that you are given the following portfolio

$$\mathcal{P} = \begin{cases} \text{short} & 1 & \text{strike-}k_1 \text{ call } (C_1) \\ \text{short} & 2 & \text{strike-}k_2 \text{ call } (C_2) \end{cases},$$

where $k_1 < k_2$, and the present value of the two options are $\hat{c}_1^0$ and $\hat{c}_2^0$, respectively.

(a) How many stock shares with present value $\hat{x}_0$ would you need to buy/sell to augment your portfolio in case you expect the market value of the stock to develop to values $X_T \in (k_1, k_2)$ at expiry? Derive the expiry Profit-Loss-profile $Z(X)$ as a function of the share price X for the augmented portfolio. Sketch Z as a function of X.

(b) Obtain the maximum profit, the maximum loss and the break even point(s) of your portfolio.

(c) Assuming that k_1, k_2, $\hat{c}_1^0$ and $\hat{c}_2^0$ are fixed quantities, provide a condition on the stock price for which you would make a profit with such a portfolio in the case $X_t \in (k_1, k_2)$.

(d) Is the portfolio you calculated risky? Explain your answer.

2. Consider a European put option with value $P(x, t; k)$ at time t with strike k, constant interest rate r and expiry time T. Let x be the present price of the stock share. Prove using arbitrage tables the following inequalities

(a)
$$P(x, t; k) \leq ke^{-r(T-t)} .$$

(b)
$$P(x, t; k) \geq ke^{-r(T-t)} - x .$$

3. Assume that a stock price follows a standard binomial process with parameters

$$x = \$10 , \qquad u = 1.1 , \qquad d = 0.9 , \qquad r = 0 .$$

An American derivative V with 2-periods to expiry written on the stock pays

$$V_T = \frac{1}{2}[X_T] + X_T \, \mathbb{I}(X_T \leq 9) .$$

Here $\mathbb{I}$ denotes the indicator function, and $[X]$ is a function which returns the nearest integer to X.

(a) Draw the stock and derivative trees to two time steps and determine the fair value V_0 of the derivative.

(b) Under which condition should the derivative be exercised early?

(c) If the derivative is currently trading for \$5 what should you do to take advantage of this mispricing? Provide exact expressions of your strategy.

4. Consider the function

$$f(\xi) = \gamma \xi^2 \mathbb{I}(\xi < a) \,,$$

where $\mathbb{I}$ denotes the indicator function.

(a) Determine γ such that $f(\xi)$ is a valid probability density function on $\mathbb{R}^+$ for $a > 0$.

(b) Assume that $f(\xi)$ is the risk-neutral measure for the stock price with current value x, expiry T. Let r be the constant interest rate. Determine γ as a function of x, r and T using the risk-neutral restriction.

(c) Calculate the fair value of a put option with $P_T = (k - X_T)^+$ using the risk-free pdf determined in parts (a) and (b). What is the limiting fair price of the put option in the limits $x \to 0$ and $x \to \infty$?

5. The price of a stock XYZ follows geometrical Brownian motion with constant volatility σ and riskless interest rate r. A derivative security written on this stock has payoff at expiry T given by

$$V_T = \frac{1}{X_T} \log(X_T/k) \,,$$

where X_T is the stock price at expiry and $k > 0$ is a constant.

Find the present value $V(x,t)$ of the derivative at any time $t < T$ given the stock price at time t is x. Express your answer in terms of $\tau := T - t$ (not t).