1. The Klein-Gordon equation is a linearised version of the Sine-Gordon equation for small \( u(x, t) \):

\[
Au_{tt} - Ku_{xx} + Tu = 0
\]

where \( A, K \) and \( T \) are constants.

(a) Find all travelling wave solutions to this equation.

(b) If \( u(x, t) \) is, indeed, small then \( u \) must remain bounded. Which wavespeeds \( c \) admit a travelling wave solution which is bounded? Sketch some representative bounded travelling wave solutions.

(c) Is the Klein-Gordon equation dispersive? In particular, do wave train solutions with high frequency travel faster, slower or at the same speed as solutions with low frequency?

(d) Show that there is a cutoff frequency \( \omega_0 \) such that solutions with frequency \( \omega \leq \omega_0 \) are not permitted.

Adapted from Knobel An Introduction to the Mathematical Theory of Waves.

2. (a) Consider the equation

\[
 u_t + c(x, t)u_x = 0.
\]

Show that along a characteristic curve \( x = x(t) \) that

\[
 \frac{d}{dt} (u(x(t), t)) = 0
\]

where \( \frac{dx}{dt} = c(x, t) \).

(b) Consider the following initial value problem:

\[
 u_t + txu_x = 0, \quad u(x, 0) = \frac{1}{1 + x^2},
\]

where \(-\infty < x < \infty\) and \( t \geq 0 \).

Question 2 continues on the next page
(i) Solve the equation \( \frac{dx}{dt} = c(x, t) \) to find the explicit solution for the family of characteristic curves. Hence show that the characteristic curve with \( x = x_0 \) when \( t = 0 \) has equation \( x = x_0e^{t^2/2} \). Plot several characteristics in the \( xt \)-plane for \( t \in [0, 2] \) and \( x \in [-5, 5] \).

(ii) Write down the solution \( u(x, t) \) obtained using the method of characteristics.

Adapted from Knobel *An Introduction to the Mathematical Theory of Waves.*