The University of Sydney
School of Mathematics and Statistics

Tutorial for Week 1

MATH3961: Metric Spaces — Semester 1, 2012

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Most of the following exercises review notation and terminology from first-year courses relating to: set X, membership $x \in X$, inclusion $X \subseteq Y$, union $X \cup Y$, intersection $X \cap Y$, complement $X \backslash Y$, function $f : X \to Y$, injective $(1-1)$, surjective (onto), bijective (invertible).

You should read §1.1 of Dr Choo's *Lecture Notes on Metric Spaces*.

We shall also need the *cartesian product* $X \times Y$ (the set of ordered pairs (x, y) with $x \in X$ and $y \in Y$), and the *power set* $\mathcal{P}(X) = 2^X = \{A \mid A \subseteq X\}$ (the set of all subsets of X), as we shall define a *metric* on a set X as a function $d : X \times X \to [0, \infty)$, and a *topology* on X as a family of subsets of X.

We may need to define *equivalence relations* later.

1. If A, B and C are subsets of X, prove that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

2. Let X be a nonempty set and A and B be two subsets of X. Prove that

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B).$$

3. In $\mathbb{R}$, let $A_1 = (-1,\ 1)$, $A_2 = (-1/2,\ 1/2)$, $\ldots$, $A_k = (-1/k,\ 1/k)$, $\ldots$. Show that $\bigcap_{k=1}^{\infty} A_k = \{0\}$.

4. Let X and Y be any two nonempty sets and let $f : X \to Y$ be any mapping. Then, for any subsets A and B of X, prove that

(i) $f(A \cup B) = f(A) \cup f(B)$

(ii) $f(A \cap B) \subseteq f(A) \cap f(B)$

(iii) $f(B \setminus A) \supseteq f(B) \setminus f(A)$

5. Let X and Y be any two nonempty sets and let $f : X \to Y$ be any mapping. Then, for any subsets A and B of Y, prove that

(i) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(ii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

(iii) $f^{-1}(B \setminus A) = f^{-1}(B) \setminus f^{-1}(A)$

6. Let X and Y be any two nonempty sets and let $f : X \to Y$ be any mapping. Then for any subset A of X and any subset B of Y, prove that

(i) $A \subseteq f^{-1}\big(f(A)\big)$, and equality holds if f is injective.

(ii) $f\big(f^{-1}(B)\big) \subseteq B$, and equality holds if f is surjective.

7. Let $\{X_i | i \in I\}$ be a family of sets (possibly with repetitions or overlaps). The *product* of this family is the set of functions f from I to $\cup_{i\in I} X_i$ such that $f(i) \in X_i$ for all $i \in I$. Notation: $\Pi_{i\in I} X_i$.

If $X_i = X$ for all $i \in I$ we also write X^I for the product.

What is the product when I has 1 member? when I has two members?

8. Let $\emptyset$ be the empty set, and let S be a nonempty set. How many members are there in each of the following sets?

(i) $S^{\emptyset}$;

(ii) $\emptyset^S$;

(iii) $\emptyset^{\emptyset}$.

9. If $0 < a \leq b$, show that $\dfrac{a}{1+a} \leq \dfrac{b}{1+b}$.