# The University of Sydney

Faculties of Arts, Economics, Education,
Engineering and Science

**MATH3966: Modules and Group Representations (Advanced)**

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Time allowed: 2 hours plus 10 minutes reading time

This booklet contains **4** pages.

This paper comprises 6 questions worth a total of 70 marks.

All questions should be attempted. All working should be shown unless the question specifies otherwise. If you can't solve one part of a question, you can still assume the result in doing later parts.

No notes, books, or calculators are allowed.

**1.** Throughout this question, $R$ denotes a principal ideal domain.

(*i*) If $\{x_1, \cdots, x_s\}$ is a subset of an $R$-module $M$, what does it mean to say that $M$ is *generated* by $\{x_1, \cdots, x_s\}$? (Give the definition.)

(*ii*) Suppose that $M$ is an $R$-module and $N$ is an $R$-submodule of $M$ such that both $N$ and $M/N$ are finitely-generated $R$-modules. Prove that $M$ is a finitely-generated $R$-module.

(*iii*) Either prove the following statement, or show it is false by giving a counter-example: if every finitely-generated $R$-module has a basis, then $R$ is a field. (You may use any results from lectures that you find useful.)

(*iv*) Suppose that $M$ is a nonzero $R$-module. What does it mean to say that $M$ is *indecomposable*? (Give the definition.)

(*v*) Prove that $R$ itself, regarded as an $R$-module, is indecomposable.

(*vi*) Give an example of a PID $R$ and a non-unit element $r \in R$ such that the $R$-module $R/Rr$ is decomposable.

**[15 marks]**

**2.** In this question, you may use any of the general results proved in lectures.

(*i*) Show that the following integer matrix has invariant factors $2, 6, 0$:

$$A = \begin{pmatrix} 0 & 2 & 6 \\ 6 & 4 & 6 \\ 6 & 2 & 0 \end{pmatrix}.$$

(*ii*) Let $M$ be the submodule of $\mathbb{Z}^3$ generated by the columns of the matrix $A$ in the previous part. Is $M$ a free $\mathbb{Z}$-module? Explain your answer.

(*iii*) Give the primary decomposition of the $\mathbb{Z}$-module $\mathbb{Z}^3/M$, where $M$ is as in the previous part. That is, state an isomorphism between $\mathbb{Z}^3/M$ and a direct sum of indecomposable $\mathbb{Z}$-modules.

(*iv*) List all the $\mathbb{Z}$-modules with 36 elements, giving one representative of each isomorphism class. (No explanation is necessary.)

**[10 marks]**

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**3.** (*i*) Find the Jordan canonical form of the rational matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 16 & -7 & -8 \\ -8 & 4 & 5 \end{pmatrix}.$$

(*ii*) Either prove the following statement, or show it is false by giving a counter-example: if $B \in \mathrm{Mat}_3(\mathbb{Q})$ has the same minimal polynomial as the matrix $A$ in the previous part, then $B$ is conjugate to $A$.

(*iii*) For any $\lambda \in \mathbb{C}$, let $J_n(\lambda)$ denote the Jordan block matrix

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \in \mathrm{Mat}_n(\mathbb{C}).$$

Prove that if $\lambda \neq 0$, $J_n(\lambda)^2$ is conjugate to $J_n(\lambda^2)$. (*Hint – one of several possible methods*: first show that $J_n(\lambda)^2$ has characteristic polynomial $(x - \lambda^2)^n$, and then it suffices to prove that this is also its minimal polynomial.)

(*iv*) Hence prove that for every invertible complex matrix $A \in GL_n(\mathbb{C})$ there is an invertible matrix $B \in GL_n(\mathbb{C})$ such that $B^2 = A$.

**[10 marks]**

**4.** In each part of this question, $G$ is a finite group and $V$ is a finite-dimensional $\mathbb{C}G$-module.

(*i*) Let $W$ be a vector subspace of $V$. State the extra condition that $W$ must satisfy in order to be a $\mathbb{C}G$-submodule of $V$.

(*ii*) Assuming that $V$ is nonzero, say what it means for $V$ to be a *simple* $\mathbb{C}G$-module (give the definition).

(*iii*) Either prove the following statement, or show it is false by giving a counter-example: if $W$ is a $\mathbb{C}G$-submodule of $V$ and $\sigma : V \to V$ is a $\mathbb{C}G$-module endomorphism, then $\sigma(W) \subseteq W$.

(*iv*) What does Schur's Lemma say about $\mathbb{C}G$-module endomorphisms of $V$ in the case that $V$ is simple? (State the result without proof.)

(*v*) Assume that $G$ is abelian. Using the result of the previous part, prove that every simple $\mathbb{C}G$-module is 1-dimensional.

(*vi*) Using any results from lectures or tutorials, prove the converse of the previous part: if every simple $\mathbb{C}G$-module is 1-dimensional, then $G$ is abelian.

**[15 marks]**

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**5.** In this question $n$ is an integer $\geq 2$, $F$ is a finite field, and $G = GL_n(F)$ is the group of $n \times n$ invertible matrices over the field $F$.

($i$) Show that the rule

$$T(g)(A) = gAg^{-1}, \text{ for all } g \in G,\ A \in \mathrm{Mat}_n(F),$$

defines a representation $T$ of $G$ on $\mathrm{Mat}_n(F)$.

($ii$) As a consequence of part ($i$), $\mathrm{Mat}_n(F)$ is an $FG$-module. Show that

$$\{A \in \mathrm{Mat}_n(F) \mid \mathrm{tr}(A) = 0\}$$

is an $FG$-submodule of $\mathrm{Mat}_n(F)$.

($iii$) Assuming that the characteristic of the field $F$ does not divide $n$, show that $\mathrm{Mat}_n(F)$ has an $FG$-submodule which is complementary to the one in part ($ii$). (*Warning*: Maschke's Theorem does not apply, because the characteristic of $F$ divides $|G|$.)

**[10 marks]**

**6.** Let $G$ be the group with presentation

$$\langle x, y \mid x^6 = 1,\ y^4 = 1,\ x^3 = y^2,\ yxy^{-1} = x^{-1} \rangle.$$

You may assume that it has 12 elements, divided into 6 conjugacy classes as follows:

$$\{1\},\ \{x, x^5\},\ \{x^2, x^4\},\ \{x^3\},\ \{y, x^2y, x^4y\},\ \{xy, x^3y, x^5y\}.$$

Hence there are 6 isomorphism classes of simple $\mathbb{C}G$-modules.

($i$) Find the complete character table of $G$, explaining all your reasoning. (*Hint:* it may be useful, though it is not necessary, to prove that there is a surjective homomorphism $G \to S_3$.)

($ii$) Using any valid method, determine which of the simple $\mathbb{C}G$-modules are defined over $\mathbb{R}$.

($iii$) Hence or otherwise, find the number of equivalence classes of representations of $G$ on $\mathbb{R}^2$.

**[10 marks]**

**This is the end of the examination paper**