8090 Semester 2, 2008

The University of Sydney

Faculties of Arts, Economics, Education, Engineering and Science

MATH3966: Modules and Group Representations (Advanced)

Lecturer: Anthony Henderson

Time allowed: 2 hours plus 10 minutes reading time

This booklet contains **4** pages.

This paper comprises 6 questions worth a total of 70 marks.

All questions should be attempted. All working should be shown unless the question specifies otherwise. If you can't solve one part of a question, you can still assume the result in doing later parts.

No notes, books, or calculators are allowed.

1. In this question, R denotes a general nonzero ring and M denotes an R-module.

(i) If $m \in M$, what does it mean to say that m is a *torsion element*? What does it mean to say that M is a *torsion module*? What does it mean to say that M is *torsion-free*? (Give the definitions.) **[3 marks]**

(ii) Give an example of a ring R such that the R-module R is not torsion-free, and list all the torsion elements in your example. **[2 marks]**

(iii) Prove that if M has a torsion-free submodule N such that M/N is torsion-free, then M is torsion-free. **[3 marks]**

(iv) Give an example of a ring R, an R-module M, and a submodule N of M such that N and M/N are both torsion modules but M is not a torsion module. (You need not prove these properties, just specify R, M, N.) **[3 marks]**

2. (i) Show that the following integer matrix has invariant factors $1, 2, 120$:

$$A = \begin{pmatrix} 1 & 0 & -7 \\ 2 & 2 & 2 \\ 4 & -18 & -52 \\ 10 & -50 & 10 \end{pmatrix}.$$

You may use any of the general results proved in lectures. **[4 marks]**

(ii) Let N be the submodule of $\mathbb{Z}^4$ generated by the columns of A. What do the invariant factors of A tell us about the relationship between a basis of $\mathbb{Z}^4$ and a basis of N? (You need only recall the statement of the general result in this particular example.) **[2 marks]**

(iii) Let M be the $\mathbb{Z}$-module generated by x_1, x_2, x_3, x_4 with defining relations

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + 10x_4 &= 0, \\ 2x_2 - 18x_3 - 50x_4 &= 0, \\ -7x_1 + 2x_2 - 52x_3 + 10x_4 &= 0. \end{aligned}$$

Using the previous part and suitable Isomorphism Theorems, prove an isomorphism between M and a direct sum of cyclic $\mathbb{Z}$-modules. **[3 marks]**

(iv) List all the $\mathbb{Z}$-modules with 200 elements, giving one representative of each isomorphism class. (No explanation is necessary.) **[3 marks]**

turn to page 3

3. In this question F denotes a field, and $\mathrm{Mat}_n(F)$ denotes the set of $n \times n$ matrices with entries in F.

(*i*) If $A \in \mathrm{Mat}_n(F)$, what are the *characteristic polynomial* and the *minimal polynomial* of A? (Give the definitions.) **[3 marks]**

(*ii*) In the case that $F = \mathbb{Q}$ and $n = 3$, find the characteristic polynomial and minimal polynomial of the matrix

$$A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix}.$$

You may use any results from lectures that you require. **[3 marks]**

(*iii*) Regard $\mathbb{Q}^3$ as a $\mathbb{Q}[x]$-module by letting x act via the matrix A in the previous part. State (with no proof required) an isomorphism between $\mathbb{Q}^3$ and a direct sum of indecomposable $\mathbb{Q}[x]$-modules. **[2 marks]**

(*iv*) In this part, consider the field $F = \mathbb{Z}_p$, where p is a prime number. Using the classification of matrices up to conjugation, find the number of conjugacy classes of 2×2 matrices over $\mathbb{Z}_p$. (This number will depend on p.) **[4 marks]**

4. In each part of this question, F is a field, G is a finite group, and V is a finite-dimensional FG-module.

(*i*) Let W be a subset of V. State the conditions that W must satisfy in order to be an FG-submodule of V. **[2 marks]**

(*ii*) Give an example of F, G, and V such that V is two-dimensional over F and is a simple FG-module, i.e. has no FG-submodules except $\{0\}$ and V. Include the definition of the representation of G on V, and the proof of simplicity, in your example. **[3 marks]**

(*iii*) Prove that if $\tau : V \to V$ is any F-linear transformation, the map $\sigma : V \to V$ defined by

$$\sigma(v) = \sum_{g \in G} g\tau(g^{-1}v)$$

is an FG-module endomorphism. **[3 marks]**

(*iv*) Prove that if V is simple, the endomorphism ring $\mathrm{End}_{FG}(V)$ is a division ring (in other words, every nonzero element has an inverse). **[3 marks]**

(*v*) For the example of F, G, V you gave in part (*ii*), determine whether the endomorphism ring $\mathrm{End}_{FG}(V)$ is a field or not. Explain carefully any general results you use. **[3 marks]**

5. In this question, G is a finite group with s conjugacy classes, $V_1, V_2, \cdots, V_s$ are a complete set of representatives for the isomorphism classes of simple finite-dimensional $\mathbb{C}G$-modules, and $\chi_1, \chi_2, \cdots, \chi_s$ are the characters of these modules (i.e. the irreducible characters of G).

(*i*) State (without proof) the formula in terms of characters for the multiplicity of V_i in a finite-dimensional $\mathbb{C}G$-module W. **[2 marks]**

(*ii*) Use the formula in the previous part to prove that the multiplicity of V_i in $\mathbb{C}G$ (the group algebra itself, regarded as a $\mathbb{C}G$-module in the obvious way) equals the dimension of V_i. **[3 marks]**

(*iii*) Prove that if $|G|$ is even, there are at least two V_i's which have odd dimension. Explain carefully any general results that you use. **[3 marks]**

6. The group D_6 has presentation $\langle x, y \,|\, x^6 = 1,\ y^2 = 1,\ yxy = x^{-1}\rangle$. You may assume that it has 12 elements, divided into 6 conjugacy classes as follows:

$$\{1\},\ \{x^3\},\ \{x, x^5\},\ \{x^2, x^4\},\ \{y, x^2y, x^4y\},\ \{xy, x^3y, x^5y\}.$$

Hence there are 6 isomorphism classes of simple $\mathbb{C}D_6$-modules.

(*i*) Determine the character table of D_6, explaining all steps. **[7 marks]**

(*ii*) Show that every simple $\mathbb{C}D_6$-module is defined over $\mathbb{R}$. You may use any general results that you require. **[3 marks]**

(*iii*) Hence or otherwise, determine the number of equivalence classes of faithful representations of D_6 on $\mathbb{R}^2$. **[3 marks]**

This is the end of the examination paper