The University of Sydney

Faculties of Arts, Economics, Education,
Engineering and Science

MATH3966: Modules and Group Representations (Advanced)

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Time allowed: 2 hours plus 10 minutes reading time

This booklet contains **9** pages.

This paper comprises 6 questions worth a total of 70 marks.

All working should be shown unless the question specifies otherwise. If you can't solve one part of a question, you can still assume the result in doing later parts.

No notes, books, or calculators are allowed.

SOLUTIONS

1. In this question, R denotes a general nonzero ring and M denotes an R-module.

(i) If $m \in M$, what does it mean to say that m is a *torsion element*? What does it mean to say that M is a *torsion module*? What does it mean to say that M is *torsion-free*? (Give the definitions.) **[3 marks]**

(ii) Give an example of a ring R such that the R-module R is not torsion-free, and list all the torsion elements in your example. **[2 marks]**

(iii) Prove that if M has a torsion-free submodule N such that M/N is torsion-free, then M is torsion-free. **[3 marks]**

(iv) Give an example of a ring R, an R-module M, and a submodule N of M such that N and M/N are both torsion modules but M is not a torsion module. (You need not prove these properties, just specify R, M, N.) **[3 marks]**

Solution:

(i) The definitions are: m is a torsion element if there is some nonzero $r \in R$ such that $rm = 0$; M is a torsion module if every $m \in M$ is a torsion element; M is torsion-free if the only torsion element in M is 0.

(ii) An example is $R = \mathbb{Z}_6$: the nonzero torsion elements are the zero divisors $2, 3, 4$.

(iii) Let m be a torsion element of M, and let $r \in R$ be a nonzero element such that $rm = 0$; we need to prove that $m = 0$. Since we have the equation $r(m+N) = 0+N$ in M/N which is torsion-free, we must have $m + N = 0 + N$, i.e. $m \in N$. But the only torsion element in N is 0, so $m = 0$ as required.

(iv) An example is $R = \mathbb{Z}_6$, $M = \mathbb{Z}_6$ itself with the usual module structure, and $N = \{0, 3\}$.

2. (i) Show that the following integer matrix has invariant factors $1, 2, 120$:

$$A = \begin{pmatrix} 1 & 0 & -7 \\ 2 & 2 & 2 \\ 4 & -18 & -52 \\ 10 & -50 & 10 \end{pmatrix}.$$

You may use any of the general results proved in lectures. **[4 marks]**

(ii) Let N be the submodule of $\mathbb{Z}^4$ generated by the columns of A. What do the invariant factors of A tell us about the relationship between a basis of $\mathbb{Z}^4$ and a basis of N? (You need only recall the statement of the general result in this particular example.) **[2 marks]**

(iii) Let M be the $\mathbb{Z}$-module generated by x_1, x_2, x_3, x_4 with defining relations

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + 10x_4 &= 0, \\ 2x_2 - 18x_3 - 50x_4 &= 0, \\ -7x_1 + 2x_2 - 52x_3 + 10x_4 &= 0. \end{aligned}$$

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Using the previous part and suitable Isomorphism Theorems, prove an isomorphism between M and a direct sum of cyclic $\mathbb{Z}$-modules. **[3 marks]**

(iv) List all the $\mathbb{Z}$-modules with 200 elements, giving one representative of each isomorphism class. (No explanation is necessary.) **[3 marks]**

Solution:

(i) Using a column operation to clear the top-right entry and then row operations to clear below the 1 in the first column, we see that A is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 16 \\ 0 & -18 & -24 \\ 0 & -50 & 80 \end{pmatrix} \text{ and thus to } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 120 \\ 0 & 0 & 480 \end{pmatrix} \text{ and thus to } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 120 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since this last matrix is in normal form, the invariant factors of A are $1, 2, 120$.

(ii) The invariant factors tell us that there is a basis $\{f_1, f_2, f_3, f_4\}$ of $\mathbb{Z}^4$ such that $\{f_1, 2f_2, 120f_3\}$ is a basis of N.

(iii) Let $\varphi : \mathbb{Z}^4 \to M$ be the $\mathbb{Z}$-module homomorphism which maps e_i to x_i. Then because of the assumed presentation, φ is surjective and the submodule N generated by the columns of A is the kernel of φ. By the First Isomorphism Theorem, $M \cong \mathbb{Z}^4/N$. With $\{f_1, f_2, f_3, f_4\}$ as in the previous part, it follows that

$$M \cong \frac{\mathbb{Z}f_1 \oplus \mathbb{Z}f_2 \oplus \mathbb{Z}f_3 \oplus \mathbb{Z}f_4}{\mathbb{Z}f_1 \oplus \mathbb{Z}2f_2 \oplus \mathbb{Z}120f_3} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{120} \oplus \mathbb{Z}.$$

(iv) Since $200 = 2^3 \times 5^2$, the possibilities are:

$$\begin{aligned}
&\mathbb{Z}_8 \oplus \mathbb{Z}_{25},\ \mathbb{Z}_8 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5, \\
&\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25},\ \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5, \\
&\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25},\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5.
\end{aligned}$$

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3. In this question F denotes a field, and $\mathrm{Mat}_n(F)$ denotes the set of $n \times n$ matrices with entries in F.

(*i*) If $A \in \mathrm{Mat}_n(F)$, what are the *characteristic polynomial* and the *minimal polynomial* of A? (Give the definitions.) **[3 marks]**

(*ii*) In the case that $F = \mathbb{Q}$ and $n = 3$, find the characteristic polynomial and minimal polynomial of the matrix

$$A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix}.$$

You may use any results from lectures that you require. **[3 marks]**

(*iii*) Regard $\mathbb{Q}^3$ as a $\mathbb{Q}[x]$-module by letting x act via the matrix A in the previous part. State (with no proof required) an isomorphism between $\mathbb{Q}^3$ and a direct sum of indecomposable $\mathbb{Q}[x]$-modules. **[2 marks]**

(*iv*) In this part, consider the field $F = \mathbb{Z}_p$, where p is a prime number. Using the classification of matrices up to conjugation, find the number of conjugacy classes of 2×2 matrices over $\mathbb{Z}_p$. (This number will depend on p.) **[4 marks]**

Solution:

(*i*) The characteristic polynomial $\chi_A(x)$ is $\det(x1 - A)$, for $x1 - A \in \mathrm{Mat}_n(F[x])$. The minimal polynomial is the unique monic polynomial $m_A(x) \in F[x]$ such that for any polynomial $p(x) \in F[x]$, $p(A) = 0$ if and only if $m_A(x) \mid p(x)$.

(*ii*) We have

$$\begin{aligned}
\chi_A(x) &= \det \begin{pmatrix} x-2 & 2 & -1 \\ -1 & x+1 & -1 \\ 0 & 1 & x+1 \end{pmatrix} \\
&= (x-2)\det \begin{pmatrix} x+1 & -1 \\ 1 & x+1 \end{pmatrix} + \det \begin{pmatrix} 2 & -1 \\ 1 & x+1 \end{pmatrix} \\
&= (x-2)(x^2+2x+2) + 2x + 3 = x^3 - 1.
\end{aligned}$$

Since the characteristic polynomial factorizes in $\mathbb{Q}[x]$ as $(x-1)(x^2+x+1)$ (distinct irreducible factors), the primary invariants of the matrix are $x-1$ and x^2+x+1. So the minimal polynomial, which is always the least common multiple of the primary invariants, is also $x^3 - 1$.

(*iii*) Since the primary invariants of A are $x - 1$ and $x^2 + x + 1$, we have

$$\mathbb{Q}^3 \cong \mathbb{Q}[x]/\mathbb{Q}[x](x-1) \,\oplus\, \mathbb{Q}[x]/\mathbb{Q}[x](x^2+x+1).$$

(*iv*) If $A \in \mathrm{Mat}_2(\mathbb{Z}_p)$, A either has a sole torsion invariant $x^2 + ax + b$ for some $a, b \in \mathbb{Z}_p$, or it has torsion invariants $x + c$ and $x + c$ for some $c \in \mathbb{Z}_p$. (There is no other possible sequence of monic polynomials whose degrees add up to 2 and which form a divisibility chain.) All values of a, b, c are possible, and each sequence of torsion invariants determines a single conjugacy class, by the classification. So the answer is $p^2 + p$.

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4. In each part of this question, F is a field, G is a finite group, and V is a finite-dimensional FG-module.

(i) Let W be a subset of V. State the conditions that W must satisfy in order to be an FG-submodule of V. **[2 marks]**

(ii) Give an example of F, G, and V such that V is two-dimensional over F and is a simple FG-module, i.e. has no FG-submodules except $\{0\}$ and V. Include the definition of the representation of G on V, and the proof of simplicity, in your example. **[3 marks]**

(iii) Prove that if $\tau : V \to V$ is any F-linear transformation, the map $\sigma : V \to V$ defined by

$$\sigma(v) = \sum_{g \in G} g\tau(g^{-1}v)$$

is an FG-module endomorphism. **[3 marks]**

(iv) Prove that if V is simple, the endomorphism ring $\mathrm{End}_{FG}(V)$ is a division ring (in other words, every nonzero element has an inverse). **[3 marks]**

(v) For the example of F, G, V you gave in part (ii), determine whether the endomorphism ring $\mathrm{End}_{FG}(V)$ is a field or not. Explain carefully any general results you use. **[3 marks]**

Solution:

(i) W must be an F-vector subspace of V which is preserved by all the representing transformations $T(g)$: that is, $gW \subseteq W$ for all $g \in G$.

(ii) An example is $F = \mathbb{C}$, $G = D_3$, and $V = \mathbb{C}^2$ with representation $T : D_3 \to GL(\mathbb{C}^2)$ defined by sending the generator x to the linear transformation with matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ and the generator y to the linear transformation with matrix $\left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$. Since the eigenvectors of $T(y)$ are (up to scalar) the vectors $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$, neither of which is an eigenvector for $T(x)$, there is no 1-dimensional subspace of $\mathbb{C}^2$ which is preserved by both $T(x)$ and $T(y)$. Hence the only $\mathbb{C}D_3$-submodules of $\mathbb{C}^2$ are $\{0\}$ and $\mathbb{C}^2$.

(iii) Since σ is defined as the sum of compositions of linear transformations, it is a linear transformation. So we need only prove that $\sigma(gv) = g\sigma(v)$ for all $g \in G$, $v \in V$. We have

$$\begin{aligned} \sigma(gv) &= \sum_{h \in G} h\tau(h^{-1}gv) \\ &= \sum_{h' \in G} gh'\tau((gh')^{-1}gv) \quad \text{(using the substitution } h = gh'\text{)} \\ &= \sum_{h' \in G} gh'\tau((h')^{-1}v) \\ &= g\sigma(v). \end{aligned}$$

(iv) Let $\varphi \in \mathrm{End}_{FG}(V)$ be nonzero. Since $\varphi : V \to V$ commutes with the action of FG, $\ker(\varphi)$ and $\mathrm{im}(\varphi)$ are FG-submodules of V. Since $\varphi \neq 0$, $\ker(\varphi)$ is not all of V; since V is simple, we have $\ker(\varphi) = \{0\}$, i.e. φ is injective. Similarly, $\mathrm{im}(\varphi)$

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is not $\{0\}$; since V is simple, we have $\mathrm{im}(\varphi) = V$, i.e. φ is surjective. So φ is an isomorphism, and has a two-sided inverse φ^{-1} in $\mathrm{End}_{FG}(V)$. Thus $\mathrm{End}_{FG}(V)$ is not a division ring.

(v) Continuing with the example $F = \mathbb{C}$, $G = D_3$, and $V = \mathbb{C}^2$ given above, every $\mathbb{C}D_3$-module endomorphism of $\mathbb{C}^2$ is a scalar multiplication by Schur's Lemma, since $\mathbb{C}^2$ is a simple $\mathbb{C}D_3$-module. So the endomorphism ring $\mathrm{End}_{\mathbb{C}D_3}(\mathbb{C}^2)$ is isomorphic to $\mathbb{C}$, and is therefore a field.

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5. In this question, G is a finite group with s conjugacy classes, $V_1, V_2, \cdots, V_s$ are a complete set of representatives for the isomorphism classes of simple finite-dimensional $\mathbb{C}G$-modules, and $\chi_1, \chi_2, \cdots, \chi_s$ are the characters of these modules (i.e. the irreducible characters of G).

(i) State (without proof) the formula in terms of characters for the multiplicity of V_i in a finite-dimensional $\mathbb{C}G$-module W. **[2 marks]**

(ii) Use the formula in the previous part to prove that the multiplicity of V_i in $\mathbb{C}G$ (the group algebra itself, regarded as a $\mathbb{C}G$-module in the obvious way) equals the dimension of V_i. **[3 marks]**

(iii) Prove that if $|G|$ is even, there are at least two V_i's which have odd dimension. Explain carefully any general results that you use. **[3 marks]**

Solution:

(i) The multiplicity equals the inner product $\langle \chi_W, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_W(g)\chi_i(g^{-1})$.

(ii) The character of $\mathbb{C}G$ is given by

$$\chi_{\mathbb{C}G}(g) = \begin{cases} |G|, & \text{if } g = 1, \\ 0, & \text{otherwise.} \end{cases}$$

So the inner product $\langle \chi_{\mathbb{C}G}, \chi_i \rangle$ is

$$\frac{1}{|G|} \sum_{g \in G} \chi_{\mathbb{C}G}(g)\chi_i(g^{-1}) = \frac{1}{|G|} |G| \chi_i(1) = \chi_i(1) = \dim V_i.$$

(iii) By the previous part and the complete reducibility of finite-dimensional $\mathbb{C}G$-modules, we know that $\mathbb{C}G$ is isomorphic to the direct sum

$$V_1 \oplus \cdots \oplus V_1 \oplus \cdots \oplus V_s \oplus \cdots \oplus V_s,$$

where each V_i occurs $\dim V_i$ times. Taking dimensions, this implies that

$$|G| = (\dim V_1)^2 + \cdots + (\dim V_s)^2.$$

Now at least one of the squares on the right-hand side is odd, because one of the simple modules is the one-dimensional trivial module. If $|G|$ is even, the remaining squares cannot all be even, or the whole sum would be odd. So there must be at least two odd squares on the right-hand side, which proves the claim.

6. The group D_6 has presentation $\langle x, y \mid x^6 = 1,\ y^2 = 1,\ yxy = x^{-1} \rangle$. You may assume that it has 12 elements, divided into 6 conjugacy classes as follows:

$$\{1\},\ \{x^3\},\ \{x, x^5\},\ \{x^2, x^4\},\ \{y, x^2y, x^4y\},\ \{xy, x^3y, x^5y\}.$$

Hence there are 6 isomorphism classes of simple $\mathbb{C}D_6$-modules.

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(i) Determine the character table of D_6, explaining all steps. **[7 marks]**

(ii) Show that every simple $\mathbb{C}D_6$-module is defined over $\mathbb{R}$. You may use any general results that you require. **[3 marks]**

(iii) Hence or otherwise, determine the number of equivalence classes of faithful representations of D_6 on $\mathbb{R}^2$. **[3 marks]**

Solution:

(i) We first find the one-dimensional characters, i.e. group homomorphisms $\psi : D_6 \to \mathbb{C}^\times$. The relation $\psi(y)^2 = 1$ implies that $\psi(y) = \pm 1$, and hence the relation $\psi(y)\psi(x)\psi(y) = \psi(x)^{-1}$ implies that $\psi(x) = \pm 1$, automatically implying the other relation $\psi(x)^6 = 1$. So there are four one-dimensional characters, which we can call $1, \psi_1, \psi_2, \psi_1\psi_2$: the group $D_6^\vee$ is isomorphic to D_2, and in fact all these one-dimensional characters factor through the surjective group homomorphism $D_6 \to D_2$ which sends x to x and y to y. The remaining two simple $\mathbb{C}D_6$-modules must both be two-dimensional, because the sum of the squares of the dimensions is $|D_6| = 12$. (We are also guaranteed that they have real values, because every conjugacy class of D_6 is self-inverse.)

One way to proceed is to consider our standard representation of D_6 on $\mathbb{R}^2$, which we can complexify to make a $\mathbb{C}D_6$-module; let χ be its character. Since x^j is represented by a transformation with eigenvalues $e^{\frac{\pm 2\pi i j}{6}}$, $\chi(x^j) = 2\cos(\frac{\pi j}{3})$, which means that

$$\chi(1) = 2,\ \chi(x) = 1,\ \chi(x^2) = -1,\ \chi(x^3) = -2.$$

Since $x^j y$ is represented by a transformation with eigenvalues 1 and -1, $\chi(y) = \chi(xy) = 0$. We can now calculate that

$$\langle \chi, \chi \rangle = \frac{2^2 + 2 \times 1^2 + 2 \times (-1)^2 + (-2)^2}{12} = 1,$$

which implies that χ is one of the two-dimensional irreducible characters. Since $\psi_1\chi$ is different from χ, it must be the other one. (Note that $\psi_2\chi = \chi$.) So the character table is:

	1	x^3	x	x^2	y	xy
	1	1	2	2	3	3
1	1	1	1	1	1	1
ψ_1	1	-1	-1	1	1	-1
ψ_2	1	1	1	1	-1	-1
$\psi_1\psi_2$	1	-1	-1	1	-1	1
χ	2	-2	1	-1	0	0
$\psi_1\chi$	2	2	-1	-1	0	0

Observe that the two-dimensional $\mathbb{C}D_6$-module with character $\psi_1\chi$ does not afford a faithful representation of D_6: it factors through the surjective group homomorphism $D_6 \to D_3$ which sends x to x and y to y, and this provides another way of finding the two-dimensional characters.

(ii) This is automatically true for the one-dimensional $\mathbb{C}D_6$-modules, since their characters are real. In the previous part we saw that a $\mathbb{C}D_6$-module V with character χ is obtained by complexifying an $\mathbb{R}D_6$-module, so it is defined over $\mathbb{R}$; it

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follows that the other simple module $\psi_1 \otimes V$ is defined over $\mathbb{R}$ also. Alternatively, a short calculation of Frobenius-Schur indicators shows that $FS(\chi) = FS(\psi_1\chi) = 1$.

(*iii*) By the previous parts, there are six simple $\mathbb{R}D_6$-modules up to isomorphism, which it is natural to call $\mathbb{R}$, $\mathbb{R}_{\psi_1}$, $\mathbb{R}_{\psi_2}$, $\mathbb{R}_{\psi_1\psi_2}$, $\mathbb{R}^2_\chi$, $\mathbb{R}^2_{\psi_1\chi}$. This gives 12 two-dimensional $\mathbb{R}D_6$-modules up to isomorphism, the two simple ones and those which are the sum of two one-dimensional ones. Since x^2 acts trivially on all the one-dimensional modules, it will act trivially on any sum of one-dimensional modules, so none of these affords a faithful representation. Of the two simple two-dimensional $\mathbb{R}D_6$-modules, we have already seen that $\mathbb{R}^2_\chi$ affords a faithful representation and $\mathbb{R}^2_{\psi_1\chi}$ does not. So there is only one faithful representation of D_6 on $\mathbb{R}^2$ up to equivalence, namely the standard representation identifying D_6 with the group of symmetries of a regular hexagon.