Solutions to Assignment 1

Solution to Exercise 1.61. Suppose that $\theta \neq 0$. Then $\ker(\theta)$ is a submodule of M which is not equal to M itself; since M is simple, $\ker(\theta) = \{0\}$, which means that θ is injective. Similarly, $\mathrm{im}(\theta)$ is a submodule of N which is not equal to $\{0\}$; since N is simple, $\mathrm{im}(\theta) = N$, which means that θ is surjective. So θ is an isomorphism.

In the case that $N = M$, this shows that any nonzero endomorphism $\theta \in \mathrm{End}_R(M)$ is invertible, so the endomorphism ring of a simple module is a division ring. (If it is also commutative, it must be a field; we will see examples later where the non-commutative division ring $\mathbb{H}$ appears.)

Solution to Exercise 1.115. Since M/N is free it has a basis, so there is some subset X of M such that $\{x + N \,|\, x \in X\}$ is a basis of M/N. Define the map $f : \{x + N \,|\, x \in X\} \to M$ by sending $x + N$ to x, for all $x \in X$. By the unique extension property of Proposition 1.78, this map can be uniquely extended to an R-module homomorphism $\psi : M/N \to M$. Every element of M/N can be written as an R-linear combination $\sum_{x\in X} r_x(x + N) = (\sum_{x\in X} r_x x) + N$, and we have

$$\psi(\sum_{x\in X} r_x(x + N)) = \sum_{x\in X} r_x\psi(x + N) = \sum_{x\in X} r_x x,$$

so ψ has the special property that $\psi(m + N) \in m + N$ for all $m \in M$, i.e. it takes each coset to an element of itself. (Such a homomorphism is called a "section" of the projection map $M \to M/N$.)

Now we seek to show that the submodule $N' = \mathrm{im}(\psi)$ of M is complementary to N. Firstly, we must show that $N \cap N' = \{0\}$: but if $n \in N \cap N'$, then $n = \psi(m + N)$ for some $m \in M$, which implies that $n \in m + N$, so $m + N = 0 + N$, so $n = \psi(0 + N) = 0$. Secondly, we must show that $M = N + N'$: for any $m \in M$, we have $m = (m - \psi(m + N)) + \psi(m + N)$, and $m - \psi(m + N) \in N$ because $\psi(m + N) \in m + N$.

Solution to Exercise 2.27. Let M be a finitely-generated R-module. Choose a generating set $\{x_1, \cdots, x_n\}$ of M; as seen in Proposition 1.71, this gives rise to a surjective R-module homomorphism $\varphi : R^n \to M$. If N is any submodule of M, then $\varphi^{-1}(N)$ is a submodule of R^n. But by Theorem 2.20, $\varphi^{-1}(N)$ must have some finite basis $\{y_1, \cdots, y_m\}$ where $m \leq n$; since φ is surjective, we have $N = \varphi(\varphi^{-1}(N))$, so N is generated by $\{\varphi(y_1), \cdots, \varphi(y_m)\}$. So we have shown that every submodule of a finitely-generated R-module is finitely-generated. (As observed in Exercise 1.60, this is not true for the ring $F[x_1, x_2, \cdots]$. The commutative rings R for which it is true are called "noetherian" after the German algebraist Emmy Noether.)

Solution to Exercise 2.47. Suppose there were three linearly independent elements $\left(\begin{smallmatrix} a \\ d \end{smallmatrix}\right), \left(\begin{smallmatrix} b \\ e \end{smallmatrix}\right), \left(\begin{smallmatrix} c \\ f \end{smallmatrix}\right)$ of R^2. The matrix $A = \left(\begin{smallmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 0 \end{smallmatrix}\right)$ clearly has determinant zero, because it has a zero row. Hence $A(\operatorname{adj} A) = 0$. Restricting to the first two rows and third column of this matrix equation, we obtain

$$(bf - ce)\left(\begin{smallmatrix} a \\ d \end{smallmatrix}\right) - (af - cd)\left(\begin{smallmatrix} b \\ e \end{smallmatrix}\right) + (ae - bd)\left(\begin{smallmatrix} c \\ f \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right).$$

By the linear independence we assumed, $bf - ce = af - cd = ae - bd = 0$. Using only $ae - bd = 0$, we obtain

$$b\left(\begin{smallmatrix} a \\ d \end{smallmatrix}\right) - a\left(\begin{smallmatrix} b \\ e \end{smallmatrix}\right) = e\left(\begin{smallmatrix} a \\ d \end{smallmatrix}\right) - d\left(\begin{smallmatrix} b \\ e \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right),$$

so linear independence again implies that $a = b = d = e = 0$. But this means that two (and similarly the third) of our linearly independent elements are actually zero, which is a contradiction.

For the general proof, it is best to prove by induction on n the statement that if $A \in \operatorname{Mat}_n(R)$ has linearly independent columns, then $\det(A) \neq 0$. (The argument is similar to the above, assuming that $\det(A) = 0$ and working from the equation $A(\operatorname{adj} A) = 0$ to reach a contradiction via the induction hypothesis.) Then if you had $n+1$ linearly independent elements of R^n, you could form an $(n + 1) \times (n + 1)$ matrix by adding an extra row, and get a contradiction.