Solutions to Assignment 2

Solution to Exercise 4.39. It is equivalent to answer the same question for matrix representations $M : D_3 \to GL_2(\mathbb{R})$. Given the standard presentation of D_3, a matrix representation $M : D_3 \to GL_2(\mathbb{R})$ is completely specified by the choice of $M(x), M(y) \in GL_2(\mathbb{R})$ satisfying

$$M(x)^3 = 1,\ M(y)^2 = 1,\ M(y)M(x)M(y) = M(x)^{-1}.$$

Two such representations $M, M' : D_3 \to GL_2(\mathbb{R})$ are equivalent if and only if there is some $A \in GL_2(\mathbb{R})$ which simultaneously satisfies $AM(x)A^{-1} = M'(x)$ and $AM(y)A^{-1} = M'(y)$ (as seen in Proposition 4.34, these equations imply $AM(g)A^{-1} = M'(g)$ for all other $g \in D_3$).

In Exercise 3.61 we saw that there are three conjugacy classes of matrices $M \in GL_2(\mathbb{R})$ such that $M^2 = 1$: the classes of the scalar matrices 1 and -1, and the class of all matrices with eigenvalues $1, -1$. Also there are two conjugacy classes of matrices $M \in GL_2(\mathbb{R})$ such that $M^3 = 1$: the class of the scalar matrix 1, and the class of all matrices with eigenvalues ω, ω^2. So we can divide the pairs $(M(x), M(y))$ into six cases, and then consider what it means to impose the third relation $M(y)M(x)M(y) = M(x)^{-1}$:

(1) $M(x) = M(y) = 1$ (third relation automatic);

(2) $M(x) = 1$, $M(y) = -1$ (third relation automatic);

(3) $M(x) = 1$, and $M(y)$ has eigenvalues $1, -1$ (third relation automatic);

(4) $M(x)$ has eigenvalues ω, ω^2, and $M(y) = 1$ (third relation fails, so this case does not arise);

(5) $M(x)$ has eigenvalues ω, ω^2, and $M(y) = -1$ (third relation fails here too);

(6) $M(x)$ has eigenvalues ω, ω^2, and $M(y)$ has eigenvalues $1, -1$.

In case 6, the relation $M(y)M(x)M(y) = M(x)^{-1}$ does need to be imposed (it doesn't follow automatically). We know that there is at least one example where it is satisfied, because our standard faithful representation $T : D_3 \to GL(\mathbb{R}^2)$, in which $T(x)$ is the anti-clockwise rotation through $\frac{2\pi}{3}$ and $T(y)$ is the reflection in the x-axis, gives a matrix representation in case 6.

It is clear that the first three cases constitute three different equivalence classes of (non-faithful) representations $M : D_3 \to GL_2(\mathbb{R})$. The claim is that case 6 also constitutes a single equivalence class, i.e. every matrix representation in case 6 is equivalent to the standard one(s) given in Example 4.26. Knowing this, the answer to the question is that there are four equivalence classes of representations of D_3 on $\mathbb{R}^2$, only one of which consists of faithful representations.

This claim amounts to saying that for any $M(x), M(y) \in GL_2(\mathbb{R})$ such that $M(x)$ has eigenvalues ω, ω^2, $M(y)$ has eigenvalues $1, -1$, and

$$M(y)M(x)M(y) = M(x)^{-1},$$

there is some $A \in GL_2(\mathbb{R})$ such that

$$AM(x)A^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \; AM(y)A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(Here we are using the first of the two equivalent matrix representations in Example 4.26, but you could use the second for a similar argument.) Now there certainly does exist some $B \in GL_2(\mathbb{R})$ satisfying just the second equation, $BM(y)B^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, since all matrices with eigenvalues $1, -1$ are conjugate to each other. Let $BM(x)B^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Conjugating the third relation by B, we see that

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which amounts to the four equations

$$a^2 - bc = 1, \; ab - bd = 0, \; -ac + cd = 0, \; -bc + d^2 = 1.$$

But also the eigenvalues of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ are ω and ω^2, which means that $a+d=-1$ and $ad-bc=1$. Putting all this together, we quickly deduce that $a=d=-\frac{1}{2}$ and $bc=-\frac{3}{4}$.

If $b=-\frac{\sqrt{3}}{2}$ and $c=\frac{\sqrt{3}}{2}$, then we are finished: B is our required conjugating matrix A. Otherwise, we have to find some $C \in GL_2(\mathbb{R})$ such that

$$C\begin{pmatrix} -\frac{1}{2} & b \\ c & -\frac{1}{2} \end{pmatrix} C^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \; C\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} C^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(Then CB will be the required conjugating matrix A.) The second equation is equivalent to saying that C is a diagonal matrix, so we just need to choose diagonal entries which make the first equation work: it turns out that $C = \left(\begin{smallmatrix} -\frac{\sqrt{3}}{2} & 0 \\ 0 & b \end{smallmatrix}\right)$ is a solution.

This method is admittedly tricky; that is partly the point. The results in Chapter 6 give a way of answering this question in a few lines.

Solution to Exercise 4.92. Suppose for a contradiction that there was an FG-submodule W' of FG such that $FG = W \oplus W'$.
Method 1. Since $\dim W = |G|-1$, we have $\dim W' = 1$, so $W' = Fv$ where $v = \sum_{h\in G} a_h h$ for some $a_h \in F$. The fact that W' is an FG-submodule means that for all $g \in G$, $gv = c_g v$ for some $c_g \in F$. So

$$\sum_{h\in G} a_h gh = c_g \sum_{h\in G} a_h h.$$

Summing the coefficients of all group elements on both sides, we find that

$$\sum_{h\in G} a_h = c_g \sum_{h\in G} a_h.$$

But by assumption $v \notin W$, so we have $\sum_{h\in G} a_h \neq 0$, and we conclude that $c_g = 1$. Hence

$$\sum_{h\in G} a_h gh = \sum_{h\in G} a_h h.$$

Equating the coefficients of g on both sides, we see that $a_1 = a_g$. This is true for all $g \in G$, so $\sum_{h\in G} a_h = |G|a_1 = 0$, which is a contradiction.

Method 2. Let x denote the element $\sum_{h\in G} h \in FG$. Note that $x \in W$, since $1+1+\cdots+1$ ($|G|$ times) equals 0 in F. Now for any $g \in G$,

$$xg = \sum_{h\in G} hg = \sum_{h'\in G} h' = x.$$

Hence for any $w = \sum_{g\in G} a_g g \in FG$,

$$xw = \sum_{g\in G} a_g xg = (\sum_{g\in G} a_g)x.$$

In particular, $xw = 0$ for all $w \in W$. Since $FG = W \oplus W'$, we can write $1 = w + w'$ for unique $w \in W$, $w' \in W'$, so

$$x = x1 = xw + xw' = xw' \in W', \text{ since } W' \text{ is an } FG\text{-submodule.}$$

Thus $x \in W \cap W'$, which contradicts the assumption that $W \cap W' = \{0\}$.

Solution to Exercise 4.106.

(i) We can prove this in the contrapositive form: supposing that V has a nontrivial FG-submodule W, we want to find a non-invertible nonzero element of $\mathrm{End}_{FG}(V)$. We already saw one in the proof of Maschke's Theorem: the projection $\pi : V \to W$ constructed there can be regarded as a map from V to V, and it is clearly nonzero and not surjective, hence non-invertible.

(ii) First note that the question didn't bother to specify the representations other than by saying they were faithful; this is because, as seen in Exercise 4.39, there is only one equivalence class of faithful representations of D_3 on $\mathbb{R}^2$, and similarly for C_3.

For C_3, we saw that the group generator x could be represented by the matrix $X = \left(\begin{smallmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{smallmatrix}\right)$. It is clear that a linear transformation σ of $\mathbb{R}^2$ belongs to $\mathrm{End}_{\mathbb{R}C_3}(\mathbb{R}^2)$ precisely when it commutes with the transformation representing x, i.e. when its matrix commutes with the matrix X. So we need to solve the equation $AX = XA$ for the unknown matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. This amounts to the following system of linear equations:

$$\begin{aligned}
-\frac{1}{2}a+\frac{\sqrt{3}}{2}b &= -\frac{1}{2}a-\frac{\sqrt{3}}{2}c,\\
-\frac{\sqrt{3}}{2}a-\frac{1}{2}b &= -\frac{1}{2}b-\frac{\sqrt{3}}{2}d,\\
-\frac{1}{2}c+\frac{\sqrt{3}}{2}d &= \frac{\sqrt{3}}{2}a-\frac{1}{2}c,\\
-\frac{\sqrt{3}}{2}c-\frac{1}{2}d &= \frac{\sqrt{3}}{2}b-\frac{1}{2}d,
\end{aligned}$$

which is easily seen to be equivalent to $b=-c$ and $a=d$. So

$$\mathrm{End}_{\mathbb{R}C_3}(\mathbb{R}^2)\cong\left\{\begin{pmatrix}a & b\\ -b & a\end{pmatrix}\mid a,b\in\mathbb{R}\right\}\cong\mathbb{C},$$

where the second isomorphism is given by $\left(\begin{smallmatrix}a & b\\ -b & a\end{smallmatrix}\right)\mapsto a+bi$ (all that requires checking is that $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)^2=\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$). As expected, the endomorphism algebra is a division ring (in this case, a field). But if we changed the field to $\mathbb{C}$, the matrix $\left(\begin{smallmatrix}a & b\\ -b & a\end{smallmatrix}\right)$ would no longer always be invertible (or zero), because the determinant a^2+b^2 could be 0 without a and b both being 0 (e.g. take $a=1$, $b=i$). In fact, $\mathrm{End}_{\mathbb{C}C_3}(\mathbb{C}^2)\cong\mathbb{C}\times\mathbb{C}$ via the isomorphism sending $\left(\begin{smallmatrix}a & b\\ -b & a\end{smallmatrix}\right)$ to $(a+bi,a-bi)$. This reflects the fact that $\mathbb{C}^2$ is the direct sum of two one-dimensional $\mathbb{C}C_3$-submodules, namely the eigenspaces of $\left(\begin{smallmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{smallmatrix}\right)$.

For the group D_3, we can represent the additional generator y by the matrix $Y=\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$. Then a linear transformation σ of $\mathbb{R}^2$ belongs to $\mathrm{End}_{\mathbb{R}D_3}(\mathbb{R}^2)$ precisely when it commutes with the transformation representing x and the transformation representing y, i.e. when its matrix commutes with both X and Y. An easy calculation shows that the commutation with Y imposes the extra condition $b=0$, so

$$\mathrm{End}_{\mathbb{R}D_3}(\mathbb{R}^2)\cong\{\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\mid a\in\mathbb{R}\}\cong\mathbb{R},$$

and similarly $\mathrm{End}_{\mathbb{C}D_3}(\mathbb{C}^2)\cong\mathbb{C}$, which is a special case of Schur's Lemma part (2).

(iii) Clearly a linear transformation $\varphi:\mathbb{H}\to\mathbb{H}$ is an $\mathbb{R}Q$-module homomorphism if and only if $\varphi(ix)=i\varphi(x)$, $\varphi(jx)=j\varphi(x)$, and $\varphi(kx)=k\varphi(x)$ for all $x\in\mathbb{H}$ (these conditions imply the analogues for $-i$, $-j$, $-k$).

Equivalently, φ is an $\mathbb{R}Q$-module endomorphism of $\mathbb{H}$ exactly when it is an $\mathbb{H}$-module endomorphism. Now such an endomorphism φ satisfies $\varphi(x) = \varphi(x1) = x\varphi(1)$ for all $x \in \mathbb{H}$, so φ is merely right multiplication by the constant $\varphi(1) \in \mathbb{H}$. Conversely, right multiplication by any $y \in \mathbb{H}$ is certainly an $\mathbb{H}$-module endomorphism. So we have a natural vector space isomorphism $\mathbb{H} \cong \mathrm{End}_{\mathbb{R}Q}(\mathbb{H})$, where y corresponds to the operation $R(y)$ of right multiplication by y. However, this is **not** an isomorphism of $\mathbb{R}$-algebras, because the multiplication is reversed: $R(yz) = R(z)R(y)$, since $x(yz) = (xy)z$ (right multiply by y first, then z). So $\mathrm{End}_{\mathbb{R}Q}(\mathbb{H})$ is naturally isomorphic to the opposite algebra $\mathbb{H}^{\mathrm{op}}$, which is defined in the same way as $\mathbb{H}$ but with the multiplication rules reversed (e.g. $ij = -k$, $ji = k$). Actually, at the cost of a slight fiddle we can get back to $\mathbb{H}$ itself, since the 'quaternionic conjugation' map $a1 + bi + cj + dk \mapsto a1 - bi - cj - dk$ gives an $\mathbb{R}$-algebra isomorphism $\mathbb{H} \xrightarrow{\sim} \mathbb{H}^{\mathrm{op}}$. In any case, we observe that $\mathrm{End}_{\mathbb{R}Q}(\mathbb{H})$ is a division algebra, so by part (i) we can conclude that $\mathbb{H}$ is a simple $\mathbb{R}Q$-module.