

Solutions to Chapter 4

Solution to Exercise 4.37. Since a linear transformation of a one-dimensional vector space must be a scalar multiplication, we can identify $GL(\mathbb{R})$ with \mathbb{R}^\times (the group of nonzero real numbers under multiplication). So finding all representations of G on \mathbb{R} is the same as finding all homomorphisms $T : G \rightarrow \mathbb{R}^\times$. Moreover, since the equation $\sigma T(g)\sigma^{-1} = T'(g)$ collapses to $T(g) = T'(g)$ in this context, two such representations are equivalent if and only if they are the same.

- (i) A homomorphism $T : C_2 = \{1, x\} \rightarrow \mathbb{R}^\times$ is determined by $T(x)$, which must be a square root of 1. So there are two: the trivial representation where $T(x) = 1$, and the non-trivial (in fact faithful) representation where $T(x) = -1$.
- (ii) Since 1 is the only cube root of 1 in \mathbb{R} , the only representation of C_3 on \mathbb{R} is the trivial one. (Note that the answer would be different if we replaced \mathbb{R} by \mathbb{C} .)
- (iii) D_2 has elements $1, x, y, xy$, and is generated by x and y subject to the defining relations $x^2 = y^2 = 1, xy = yx$. So a homomorphism $T : D_2 \rightarrow \mathbb{R}^\times$ is determined by the choice of $T(x)$ and $T(y)$, each of which must be ± 1 ; the third relation is automatic because numbers commute. So there are four such homomorphisms, whose values are given in the rows of the following table:

	1	x	y	xy
	1	1	1	1
	1	-1	1	-1
	1	1	-1	-1
	1	-1	-1	1

Notice the symmetry in the three non-trivial homomorphisms, reflecting the fact that the group presentation would be exactly the same if we used $u = x$, $v = xy$ as the generators, or $u = xy$, $v = y$, etc. Also notice that none of these four is a faithful representation of D_2 .

- (iv) D_3 has elements $1, x, x^2, y, xy, x^2y$, and is generated by x and y subject to the defining relations $x^3 = y^2 = 1$, $xyx = x^{-1}$. So a homomorphism $T : D_3 \rightarrow \mathbb{R}^\times$ is determined by the choice of real numbers $T(x)$ and $T(y)$ satisfying

$$T(x)^3 = 1, T(y)^2 = 1, T(y)T(x)T(y) = T(x)^{-1}.$$

Now the first equation says that $T(x) = 1$, and the second and third equations both say that $T(y) = \pm 1$. Thus there are two homomorphisms $D_3 \rightarrow \mathbb{R}^\times$, the trivial one and the one where $T(y) = -1$. The values of this second homomorphism are:

$$\begin{array}{cccccc} 1 & x & x^2 & y & xy & x^2y \\ \hline 1 & 1 & 1 & -1 & -1 & -1 \end{array}$$

If we identify D_3 with the group S_3 of permutations of $\{1, 2, 3\}$ as in Example 4.51, this homomorphism just gives the sign of each permutation (1 for even permutations, -1 for odd permutations).

Solution to Exercise 4.38.

- (i) All we have to check is that the given matrices satisfy the relations in the presentation of D_3 ; they then extend to a well-defined matrix representation $D_3 \rightarrow GL_3(\mathbb{R})$, which is equivalent to a linear representation U as required. So it is just a matter of matrix multiplication:

$$\begin{aligned} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^3 &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

A more conceptual construction of this representation is given in Example 4.57.

- (ii) It suffices to show that P is preserved by $U(x)$ and $U(y)$, since x and y generate D_3 . But

$$U(x) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_3 \\ a_1 \\ a_2 \end{pmatrix}, \quad U(y) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_3 \\ a_2 \end{pmatrix}.$$

So both $U(x)$ and $U(y)$ preserve the sum $a_1 + a_2 + a_3$, and hence preserve P .

- (iii) It suffices to find a basis of P with respect to which the representing matrices of the generators are

$$[U'(x)] = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad [U'(y)] = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},$$

since these are representing matrices for T as seen in Example 4.26 (and if the matrices coincide for the generators, they must coincide for all elements of the group). To do this, note that the first basis element is supposed to be a 1-eigenvector of $U'(y)$. It is easy to see that the 1-eigenspace of $U(y)$ in \mathbb{R}^3 is the span of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, so we have to choose an element of this span which lies in P , say $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, to be the first basis vector. Then $U'(x)$ is meant to map this first basis vector to the second basis vector, which we must therefore choose to be $U(x)\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. Certainly $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ form a basis of P , and all that remains to check is that the action of $U'(x)$ and $U'(y)$ on the second basis vector is what it should be. But

$$U'(x) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = U'(y) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = - \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

as required.

Solution to Exercise 4.39. Part of Assignment 2.

Solution to Exercise 4.40. A 2×2 matrix is invertible if and only if the columns are linearly independent, which over \mathbb{Z}_2 just means that they are nonzero and not equal. So there are six matrices in $GL_2(\mathbb{Z}_2)$, namely

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The identity (of order 1) is alone in its conjugacy class. The elements of order 2, namely $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, are conjugate because the third is the rational canonical form of the first two (they all have characteristic polynomial $x^2 + 1 = (x + 1)^2$ which is also their minimal polynomial). The elements of order 3, namely $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, are conjugate because the second is the rational canonical form of the first (both have characteristic polynomial $x^2 + x + 1$, which is irreducible in $\mathbb{Z}_2[x]$). This is all the elements, so there are just these three conjugacy classes.

A matrix representation $M : D_3 \rightarrow GL_2(\mathbb{Z}_2)$ is completely specified by the choice of two elements $M(x), M(y) \in GL_2(\mathbb{Z}_2)$ which satisfy

$$M(x)^3 = 1, \quad M(y)^2 = 1, \quad M(y)M(x)M(y) = M(x)^{-1}.$$

The possible cases are:

- (1) $M(x) = M(y) = 1$ (the trivial representation);
- (2) $M(x) = 1$, $M(y)$ has order 2 (the third relation is automatic);
- (3) $M(x)$ has order 3, $M(y) = 1$ (the third relation fails, so this case does not occur);
- (4) $M(x)$ has order 3, $M(y)$ has order 2, and the third relation holds.

Clearly the first two cases give rise to two equivalence classes of (non-faithful) matrix representations.

To analyse the last case, it helps to prove that any pair $(X, Y) \in GL_2(\mathbb{Z}_2)$ such that X has order 3 and Y has order 2 is 'simultaneously' conjugate to the pair consisting of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, i.e. there exists $g \in GL_2(\mathbb{Z}_2)$ such that $X = g\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}g^{-1}$ and $Y = g\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}g^{-1}$. In other words, these pairs (X, Y) form a single orbit under the conjugation action of $GL_2(\mathbb{Z}_2)$. Since there

are six such pairs, this will follow from the orbit-stabilizer relation once we show that the stabilizer of the pair $((\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}), (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}))$ is trivial. This stabilizer consists of all invertible matrices which commute with both $(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix})$ and $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$, and it is easy to check that the identity is the only such matrix. This fact means that after checking that the third relation holds for $M(x) = (\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix})$ and $M(y) = (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$, which is a simple calculation, we know that it holds for any $M(x)$ of order 3 and $M(y)$ of order 2; moreover, the six resulting matrix representations are all equivalent. It is straightforward to check that one of these representations (and hence any of them) is a bijective map, so these six representations are in fact six isomorphisms $D_3 \xrightarrow{\sim} GL_2(\mathbb{Z}_2)$. In particular, they are faithful representations.

Solution to Exercise 4.41. It is very easy to see that $S(g)(\psi)$ is again a linear map from V to W , and that each $S(g)$ is a linear transformation of the vector space $\text{Hom}_F(V, W)$. By definition, we need to prove that each $S(g)$ is invertible and that $S(gh) = S(g)S(h)$ for all $g, h \in G$; but since $S(1)$ is clearly the identity, the invertibility follows from the special case $h = g^{-1}$ of the equation $S(gh) = S(g)S(h)$, so that equation is all we actually need. This follows from the corresponding equation for T and U in a natural way:

$$\begin{aligned} (S(g)(S(h)(\psi)))(v) &= U(g)((S(h)(\psi))(T(g)^{-1}(v))) \\ &= U(g)(U(h)(\psi(T(h)^{-1}(T(g)^{-1}(v)))))) \\ &= U(gh)(\psi(T(gh)^{-1}(v))) = (S(gh)(\psi))(v). \end{aligned}$$

Solution to Exercise 4.58. To check that this defines an action, we need to observe that 1 acts trivially, because $1.h = 1h1^{-1} = h$, and that the composition of two actions is the action of the product, because

$$(g_1g_2).h = (g_1g_2)h(g_1g_2)^{-1} = g_1g_2hg_2^{-1}g_1^{-1} = g_1(g_2.h)g_1^{-1} = g_1.(g_2.h).$$

The orbit of an element $h \in G$ is $\{ghg^{-1} \mid g \in G\}$, i.e. the conjugacy class of h . The stabilizer of $h \in G$ is $\{g \in G \mid gh = hg\}$, which is a subgroup called the centralizer of h , often written $Z_G(h)$. Coincidentally, this is also the fixed-point set of h (when h is viewed as a group element rather than a set element). The orbit-stabilizer relation in this case tells us that (when G is finite) the size of the conjugacy class of h is $\frac{|G|}{|Z_G(h)|}$; this has the useful consequence that the size of any conjugacy class must divide the order of

the group. Burnside's Lemma tells us that (when G is finite) the number of conjugacy classes is $\frac{1}{|G|} \sum_{h \in G} |Z_G(h)|$.

Solution to Exercise 4.59. By Exercise 4.58, a group acts on any of its conjugacy classes, by conjugation. Now two elements of S_n are conjugate if and only if they have the same cycle-type (i.e. when they are written as products of disjoint cycles, the lengths of those cycles are the same). One of the conjugacy classes of S_4 , say X , consists of all the elements which are the products of disjoint 2-cycles, namely $(12)(34)$, $(13)(24)$, and $(14)(23)$. Identify X with $\{1, 2, 3\}$ by numbering these elements in the given order. Then the action of S_4 on X by conjugation is translated into an action of S_4 on $\{1, 2, 3\}$; such an action gives rise to (indeed is equivalent to) a homomorphism $\varphi: S_4 \rightarrow S_3$. For instance, to find $\varphi((12))$, we first need to work out the action of (12) on X :

$$\begin{aligned}(12)[(12)(34)](12)^{-1} &= (12)(34), \\ (12)[(13)(24)](12)^{-1} &= (14)(23), \\ (12)[(14)(23)](12)^{-1} &= (13)(24).\end{aligned}$$

Hence (12) fixes the first element of X and swaps the other two, which means that $\varphi((12)) = (23) \in S_3$. Similarly for (23) and (34) :

$$\begin{aligned}(23)[(12)(34)](23)^{-1} &= (13)(24), & (34)[(12)(34)](34)^{-1} &= (12)(34) \\ (23)[(13)(24)](23)^{-1} &= (12)(34), & (34)[(13)(24)](34)^{-1} &= (14)(23) \\ (23)[(14)(23)](23)^{-1} &= (14)(23), & (34)[(14)(23)](34)^{-1} &= (13)(24).\end{aligned}$$

So $\varphi((23)) = (12)$ and $\varphi((34)) = (23)$. Since the image of φ contains the elements (12) and (23) which generate S_3 , $\varphi: S_4 \rightarrow S_3$ is surjective. Note that $|\ker(\varphi)| = |S_4|/|S_3| = 4$. Moreover, $\varphi((12)(34)) = \varphi((12))\varphi((34)) = (23)(23) = 1$, so $(12)(34)$ and hence all its conjugates must lie in the normal subgroup $\ker(\varphi)$. So $\ker(\varphi)$ consists of the identity and the elements of X (this group is isomorphic to D_2). Composing $\varphi: S_4 \rightarrow S_3$ with the standard homomorphism $S_3 \rightarrow GL_3(\mathbb{R})$, we get a matrix representation $S_4 \rightarrow GL_3(\mathbb{R})$ in which the generators of S_4 have the following representing matrices:

$$(12) : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (23) : \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (34) : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Clearly this representation is not faithful.

Solution to Exercise 4.60.

(i) It is geometrically clear that G acts transitively on the set V of vertices, i.e. given any two vertices there is a rotation taking one to the other. For any vertex v , the stabilizer G_v is cyclic of order 3, its elements cyclically permuting the three adjacent vertices. So by the orbit-stabilizer relation, $|G| = |G_v||V| = 3 \times 8 = 24$. (Entirely analogous arguments using faces or edges would give the same result.) Alternatively, it is not hard to count the rotations in G by classifying them into five different kinds:

- the identity;
- two 90° rotations about each of the three face axes (lines joining centres of opposite faces);
- one 180° rotation about each face axis;
- two 120° rotations about each of the four main diagonals (lines joining opposite vertices); and
- one 180° rotation about each of the six edge axes (lines joining midpoints of opposite edges).

As you would expect, these are the five conjugacy classes in G .

(ii) Clearly any rotation which preserves the cube must send a diagonal to another diagonal. So G acts on the set of diagonals. Numbering the diagonals, we can view this as an action on the set $\{1, 2, 3, 4\}$, which gives rise to a homomorphism $G \rightarrow S_4$. Since $|G| = |S_4| = 24$, we can prove that this is an isomorphism merely by showing that it is injective. But if $g \in G$ is in the kernel, then g preserves every diagonal, i.e. for every vertex v , either $gv = v$ or $gv = -v$. The only linear transformation which sends every vertex to its opposite is $-1_{\mathbb{R}^3}$, which is not a rotation (it is orientation-reversing), so we must have $gv = v$ for some vertex v . Then g must preserve the set of vertices adjacent to v , and so it must fix each of these as well, and hence g is the identity as required.

To be specific about the isomorphism $G \cong S_4$, we first have to choose the numbering of the diagonals on which this depends: say the first is the one through $(1, 1, 1)$, the second is the one through $(-1, 1, 1)$, the third is the one through $(1, -1, 1)$, and the fourth is the one through $(1, 1, -1)$. Experimenting with a model of the cube shows that, for

example, the 180° rotation about the edge axis $\mathbb{R}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ swaps the first two diagonals and preserves the other two, so it corresponds to the permutation (12). The correspondence works as follows on conjugacy classes:

- the identity rotation corresponds to the identity permutation;
 - the 90° rotations about face axes correspond to the 4-cycles;
 - the 180° rotations about face axes correspond to the products of disjoint 2-cycles;
 - the 120° rotations about main diagonals correspond to the 3-cycles; and
 - the 180° rotations about edge axes correspond to the 2-cycles.
- (iii) The inverse of the isomorphism $G \xrightarrow{\sim} S_4$ found in the previous part is an isomorphism $S_4 \xrightarrow{\sim} G$, which we can compose with the inclusion $G \subseteq GL(\mathbb{R}^3)$ to get a faithful representation of S_4 on \mathbb{R}^3 , i.e. a faithful matrix representation $S_4 \rightarrow GL_3(\mathbb{R})$. Using the specific numbering of the diagonals given in the previous part, the image of (12), for example, is the 180° rotation about the line $\mathbb{R}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. It is straightforward to find the matrix of this rotation (bear in mind that on the perpendicular plane it must act as the half-turn, i.e. multiplication by -1). The result of this and similar calculations is the following set of representing matrices for the generators:

$$(12) : \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (23) : \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (34) : \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Notice how similar these matrices are to those in Exercise 4.59; and yet this representation, being faithful, is clearly not equivalent to the previous one. For obvious reasons, this is called the ‘rotation representation’ of S_4 on \mathbb{R}^3 .

Solution to Exercise 4.89.

- (i) By definition, $V^G = \{v \in V \mid gv = v, \forall g \in G\}$. This is a vector subspace of V , because if $v, v' \in V^G$, then $g(v+v') = gv + gv' = v + v'$

for all $g \in G$, and $g(av) = agv = av$ for all $g \in G$ and $a \in F$. It is trivial that V^G is stable under the action of every $g \in G$, so V^G is an FG -submodule. Note that the representation of G on V^G is trivial (and what's more, V^G is the largest possible FG -submodule of V with that property).

(ii) Rephrasing the definition of Exercise 4.41 in module notation, we have

$$(g\psi)(v) = g\psi(g^{-1}v), \text{ for all } g \in G, \psi \in \text{Hom}_F(V, W), v \in V.$$

So for $\psi \in \text{Hom}_F(V, W)$, the condition $g\psi = \psi$ is equivalent to saying that $g\psi(g^{-1}v) = \psi(v)$ for all $v \in V$, which is in turn equivalent to saying that $g\psi(v) = \psi(gv)$ for all $v \in V$, which by definition is equivalent to saying that ψ is an FG -module homomorphism. Thus $\text{Hom}_F(V, W)^G = \text{Hom}_{FG}(V, W)$.

Solution to Exercise 4.90.

(i) We clearly have

$$(12)v_1 = v_1, (12)v_2 = -v_3, (12)v_3 = -v_2.$$

This tells us the representing matrix for (12), and the others are calculated similarly:

$$(12) : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, (23) : \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, (34) : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The matrix representation $S_4 \rightarrow GL_3(\mathbb{R})$ defined by this mapping is easily seen to be faithful, because there is no non-trivial element of S_4 which fixes every element of V . (If an element of S_4 fixed every element of V it would have to fix every element of \mathbb{R}^4 , because we know it acts trivially on the complementary line.) Hence this representation is not equivalent to the non-faithful representation in Exercise 4.59. It is also not equivalent to the rotation representation of Exercise 4.60, because the matrix representing (12) here is not conjugate to the matrix representing (12) in the rotation representation (check their determinants).

- (ii) If we let σ be the linear transformation of \mathbb{R}^3 which is the image of (12) under this representation, then from the matrix of σ we see that it has eigenvalues 1 (repeated) and -1 : its 1-eigenspace is the plane P spanned by $(1, 0, 0)$ and $(0, 1, -1)$, and its (-1) -eigenspace is the line L spanned by $(0, 1, 1)$. Since L is orthogonal to P , σ is just the reflection in the plane P . Now P is the plane of symmetry of the tetrahedron which passes through the edge joining the vertices $(1, 1, -1)$ and $(1, -1, 1)$ (and the midpoint of the opposite edge), so σ is a symmetry of the tetrahedron. Similarly one can show that (23) and (34) also act by reflection symmetries of the tetrahedron. Since these are generators for S_4 , the image of the representation is a subgroup of the group H of all symmetries of the tetrahedron. Reasoning as in Exercise 4.60(i) we can show that $|H| = 24$ (the stabilizer of a vertex is isomorphic to D_3). So the representation in part (i) identifies S_4 with the group H . Since transpositions map to reflections, this is known as the ‘reflection representation’ of S_4 on \mathbb{R}^3 .

Note that not every element of H is a reflection: if you multiply two reflections you get an orientation-preserving isometry of \mathbb{R}^3 , i.e. a rotation. Every rotation which preserves the tetrahedron must also preserve the cube in Exercise 4.60, because the eight vertices of the cube are just the four vertices of the tetrahedron and their negatives. So the group of rotations which preserve the tetrahedron is $G \cap H$, where G is as in Exercise 4.60. It is easy to see that $|G \cap H| = 12$. The corresponding subgroup of S_4 is the alternating group A_4 consisting of even permutations, and in fact the rotation and reflection representations we have defined restrict to the same representation of A_4 on \mathbb{R}^3 ; to see this, just note that for all $g \in S_4$, the two matrices representing g differ by multiplication by the sign of g , since this is true for the generators.

The group K of all symmetries of the cube has order 48 (24 rotations and 24 orientation-reversing isometries). We have seen that there are two subgroups of K which are isomorphic to S_4 , namely G (the 24 rotations) and H (the symmetries preserving the chosen tetrahedron). These subgroups are not conjugate in K .

Solution to Exercise 4.91.

- (i) No, although they are both non-abelian groups of order 8. Perhaps

the most obvious difference is in the orders of the elements: Q has one element of order 1, one element of order 2, and six elements of order 4, whereas D_4 has one element of order 1, five elements of order 2, and two elements of order 4.

- (ii) First note that there is indeed a map $T : Q \rightarrow GL(\mathbb{H})$ with this definition, since if $g \in Q$, left multiplication by g is an \mathbb{R} -linear map from \mathbb{H} to itself whose inverse is left multiplication by g^{-1} . Then all we need to check is that $T(gh) = T(g)T(h)$, which just says that $(gh)v = g(hv)$, a consequence of the associativity of multiplication in the quaternion algebra \mathbb{H} .
- (iii) The representing matrices for T are:

$$[T(1)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad [T(i)] = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$[T(j)] = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad [T(k)] = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and the negatives of these, since $T(-g) = -T(g)$ for all $g \in Q$. (By the way, this last fact is not true for every representation of Q .)

- (iv) The most obvious way to regard \mathbb{H} as a complex vector space is to let scalar multiplication by i be simply the left multiplication by the element $i \in \mathbb{H}$. If we use this, however, we will run into the immediate problem that $T(g)$ as defined in part (ii) is not always a \mathbb{C} -linear map: for instance, $iT(j)(1) = ij = k$ whereas $T(j)(i1) = ji = -k$. So the correct thing to do is to regard \mathbb{H} as a complex vector space via **right** multiplication: we had better write scalar multiplication with a dot, so $i.v$ actually means vi . We then have that \mathbb{H} is a two-dimensional complex vector space with basis consisting of 1 and j (confusingly, k is $(-i).j$). Moreover, $T(g)$ is \mathbb{C} -linear for all $g \in Q$, since $g(vi) = (gv)i$ by associativity of \mathbb{H} . The proof that T is a representation over \mathbb{C} is the same as over \mathbb{R} . Relative to the basis $1, j$, the representing matrices

are:

$$[T(1)] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, [T(i)] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$$[T(j)] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, [T(k)] = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

and the negatives of these, since it is still true that $T(-g) = -T(g)$ for all $g \in Q$. In fact, the \mathbb{R} -span of these four complex matrices is commonly used as a definition of \mathbb{H} .

Solution to Exercise 4.92. Part of Assignment 2.

Solution to Exercise 4.104. As with the linearization of any permutation representation, there are two obvious $\mathbb{R}C_3$ -submodules of $\mathbb{R}C_3$: the one-dimensional submodule $W = \mathbb{R}(1 + x + x^2)$ and the two-dimensional submodule

$$W' = \{a_1 1 + a_x x + a_{x^2} x^2 \mid a_1 + a_x + a_{x^2} = 0\}.$$

Clearly $W \cap W' = \{0\}$, so $\mathbb{R}C_3 = W \oplus W'$. It only remains to determine whether W' is simple: if $U : C_3 \rightarrow GL(W')$ is the representation by which C_3 acts on W' , this amounts to asking whether $U(x)$ has an eigenvector in W' . With respect to the basis $1 - x, x - x^2$ of W' , the representing matrix $[U(x)]$ is $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ whose eigenvalues are ω and ω^2 , so $U(x)$ does not have an eigenvector in the real vector space W' , which means it is a simple $\mathbb{R}C_3$ -module, and we have answered the question. If we work over \mathbb{C} , the two-dimensional submodule is no longer simple, and we can write $\mathbb{C}C_3$ as $\mathbb{C}(1 + x + x^2) \oplus \mathbb{C}(1 + \omega x + \omega^2 x^2) \oplus \mathbb{C}(1 + \omega^2 x + \omega x^2)$, the direct sum of three one-dimensional submodules.

Solution to Exercise 4.105.

- (i) An example with this property is the $\mathbb{R}D_2$ -module $\mathbb{R}D_2$, which we saw in Example 4.87 can be written as the direct sum of four one-dimensional submodules:

$$\mathbb{R}D_2 = \mathbb{R}(1 + x + y + xy) \oplus \mathbb{R}(1 - x + y - xy)$$

$$\oplus \mathbb{R}(1 + x - y - xy) \oplus \mathbb{R}(1 - x - y + xy).$$

None of the one-dimensional representations of D_2 is faithful, but its regular representation on $\mathbb{R}D_2$ certainly is.

- (ii) A rather trivial example with this property is when $G = \{1\}$: each of the V_i must be one-dimensional, but there are certainly subspaces of V which are not direct sums of some of these fixed one-dimensional subspaces. A less trivial example is the direct sum decomposition of $\mathbb{R}D_3$ found in Example 4.103: you can check that there are other two-dimensional submodules than the ones found there, for example the span of $-1+x-y+x^2y$ and $-x+x^2+y-xy$. This phenomenon occurs whenever two or more of the simple V_i 's are isomorphic FG -modules.
- (iii) The $\mathbb{R}C_2$ -module \mathbb{R}^2 where x is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is an example, because we can write $\mathbb{R}^2 = \mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and these two eigenlines of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the only one-dimensional submodules.

Solution to Exercise 4.106. Part of Assignment 2.