Solutions to Chapter 5

Solution to Exercise 5.11. By the classification of finite abelian groups, we may assume that $G = C_{n_1} \times \cdots \times C_{n_k}$. Identify each cyclic group C_{n_i} with the subgroup $\{(1, \cdots, 1, x, 1, \cdots, 1) \,|\, x \in C_{n_i}\}$ of G in the usual way. Let x_i be a generator of C_{n_i}. Then G has presentation

$$\langle x_1, \cdots, x_k \,|\, x_1^{n_1} = \cdots = x_k^{n_k} = 1,\, x_i x_j = x_j x_i,\, \forall i \neq j \rangle.$$

Hence any group homomorphism $\psi : G \to \mathbb{C}^\times$ is completely determined by its values $\psi(x_1), \cdots, \psi(x_k)$, and each $\psi(x_i)$ can be any n_ith root of 1. This gives a bijection $\psi \mapsto (\psi(x_1), \cdots, \psi(x_k))$ between $G^\vee$ and the group $\mu_{n_1} \times \cdots \times \mu_{n_k}$, where μ_n denotes the group of complex nth roots of 1. It is trivial to check that this bijection respects multiplication, so it is a group isomorphism. Since $\mu_n \cong C_n$, we have $G^\vee \cong G$ as required.

Solution to Exercise 5.12. If $\psi : G \to \mathbb{C}^\times$ is a one-dimensional character, then $\psi([g, h]) = [\psi(g), \psi(h)] = 1$ since $\mathbb{C}^\times$ is abelian. So $\psi(g)$ depends only on the coset $g[G, G]$, and hence ψ factors through a one-dimensional character of $G/[G, G]$. Conversely, it is clear that any one-dimensional character of $G/[G, G]$ can be pulled back to a one-dimensional character of G (by composing with the quotient map $G \to G/[G, G]$). Hence we have a bijection $(G/[G, G])^\vee \cong G^\vee$, which is clearly a group isomorphism. By Exercise 5.11, $(G/[G, G])^\vee \cong G/[G, G]$, so we have $G^\vee \cong G/[G, G]$ as required.

In particular, the groups G for which $G^\vee$ is trivial, i.e. the trivial character is the only one-dimensional character of G, are precisely those for which $[G, G] = G$. Every non-abelian simple group must have this property; there are also non-simple groups G for which $[G, G] = G$.

Solution to Exercise 5.13.

(i) We just need to observe that for all $g, h \in G$,

$$\begin{aligned}(\psi T)(g)(\psi T)(h) &= \psi(g)T(g)\psi(h)T(h)\\ &= \psi(g)\psi(h)T(g)T(h) = \psi(gh)T(gh) = (\psi T)(gh).\end{aligned}$$

(ii) A vector subspace of V is a $\mathbb{C}G$-submodule of V if and only if it is preserved by all the linear transformations $T(g)$. Since $(\psi T)(g)$ is just a nonzero scalar multiple of $T(g)$, this is equivalent to saying that V is preserved by all the linear transformations $(\psi T)(g)$, i.e. that V is a $\mathbb{C}G$-submodule of $\psi \otimes V$. So the submodules of V and $\psi \otimes V$ are actually the same subspaces, and in particular, one is simple if and only if the other is simple.

Solution to Exercise 5.24. The fact that $\chi_P(1) = 2$ shows that all the representing matrices are 2×2, so we are dealing with a matrix representation $M : D_3 \to GL_2(\mathbb{C})$. We know that $M(1)$ is the identity, so its eigenvalues are 1 and 1.

Let the eigenvalues of $M(x)$ be λ and μ. Then

$$\lambda + \mu = \chi_P(x) = -1 \text{ and } \lambda^2 + \mu^2 = \chi_P(x^2) = -1.$$

Hence $\lambda\mu = \frac{(-1)^2-(-1)}{2} = 1$, so λ and μ are the roots of the quadratic $x^2 + x + 1$, i.e. ω and ω^2. Alternatively, since $M(x)^3 = M(x^3) = 1$, λ and μ have to be cube roots of 1, and the only way to write -1 as the sum of two cube roots of 1 is as $\omega + \omega^2$. It follows immediately that the eigenvalues of $M(x^2) = M(x)^2$ are $\omega = (\omega^2)^2$ and ω^2 also.

In the case of $M(y)$, $M(xy)$, and $M(x^2y)$, we know that the eigenvalues must be ± 1 since these elements have order 2; since the traces are all zero, the eigenvalues must be 1 and -1 in each case. Of course, all these eigenvalues agree with what we saw in Example 4.26.

Solution to Exercise 5.25.

(i) Recall that $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is the vector space of linear functions $f : V \to \mathbb{C}$, which is made into a $\mathbb{C}G$-module by setting $(gf)(v) = f(g^{-1}v)$. By easy linear algebra, V is isomorphic as a vector space to $(V^*)^*$ via the map sending v to the linear function $V^* \to \mathbb{C} : f \mapsto f(v)$. A simple check shows that this is a $\mathbb{C}G$-module isomorphism. Hence it suffices to prove one direction of the statement, say that the simplicity of V^* implies the simplicity of V.

So assume that V^* is simple, and let W be a $\mathbb{C}G$-submodule of V. By easy linear algebra again, we have an orthogonal subspace $W^\perp \subseteq V^*$ defined by $W^\perp = \{f \in V^* \mid f(w) = 0, \forall w \in W\}$. It is trivial to check that $W^\perp$ is a $\mathbb{C}G$-submodule of V^*. Hence we either have $W^\perp = \{0\}$, which forces $W = V$, or $W^\perp = V^*$, which forces $W = \{0\}$. So V is simple. (Note that this part would have worked over any field, not just $\mathbb{C}$.)

(ii) Choose a basis $v_1, \cdots, v_n$ of V, and let $(a_{ij}(g))$ denote the representing matrix of g relative to this basis, so that

$$gv_j = \sum_{i=1}^{n} a_{ij}(g)v_i, \text{ for all } 1 \le j \le n.$$

Let $v_1^*, \cdots, v_n^*$ be the basis of V^* which is dual to the basis $v_1, \cdots, v_n$, in the sense that $v_i^*(v_k) = \delta_{ik}$. We need to calculate how g acts on these dual basis elements:

$$(gv_j^*)(v_l) = v_j^*(g^{-1}v_l) = v_j^*(\sum_{k=1}^{n} a_{kl}(g^{-1})v_k) = a_{jl}(g^{-1}),$$

which means that $gv_j^* = \sum_{i=1}^{n} a_{ji}(g^{-1})v_i^*$. Hence the representing matrix of g relative to the basis $v_1^*, \cdots, v_n^*$ of V^* is $(a_{ji}(g^{-1}))$, the transpose of the representing matrix of g^{-1} relative to the basis $v_1, \cdots, v_n$ of V. Since the operation of transposing leaves the trace unchanged, we have $\chi_{V^*}(g) = \chi_V(g^{-1})$. This much would work over any field; over the complex numbers, we also know that $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ by Proposition 5.18(6), so the character of the dual space χ_{V^*} is the complex conjugate of χ_V.

Solution to Exercise 5.47. To find the character of the linearization of a group action, we need to find the number of fixed points of each group element. In other words, we need to know, for any element g of S_3, how many elements h of S_4 satisfy $ghg^{-1} = h$, i.e. how many elements of S_4 commute with it. For $g = 1$ the answer is all elements of S_4, so the number is 24. For $g = (123)$, the elements of S_4 which commute with it are just 1, (123), and (132), so the number is 3; to see that there are no others, note that the conjugacy class of 3-cycles in S_4 has eight elements, so by the orbit-stabilizer relation each has a centralizer of order $24/8 = 3$. For $g = (12)$, the same argument shows that the centralizer in S_4 has order $24/6 = 4$; the elements are 1, (12), (34), and (12)(34). So the character $\chi_{\mathbb{C}S_4} : S_3 \to \mathbb{C}$ is:

	1	(12)	(123)
$\chi_{\mathbb{C}S_4}$	24	4	3

To determine the multiplicities of the simple $\mathbb{C}S_3$-modules, we must calculate the inner products of this character with the irreducible characters 1, ε, ϱ:

$$\begin{aligned}\langle \chi_{\mathbb{C}S_4}, 1\rangle &= \frac{24 \times 1 + 3 \times 4 \times 1 + 2 \times 3 \times 1}{6} = 7,\\ \langle \chi_{\mathbb{C}S_4}, \varepsilon\rangle &= \frac{24 \times 1 + 3 \times 4 \times (-1) + 2 \times 3 \times 1}{6} = 3,\\ \langle \chi_{\mathbb{C}S_4}, \varrho\rangle &= \frac{24 \times 2 + 3 \times 4 \times 0 + 2 \times 3 \times (-1)}{6} = 7.\end{aligned}$$

Hence the trivial one-dimensional $\mathbb{C}S_3$-module occurs in $\mathbb{C}S_4$ with multiplicity 7, the non-trivial one-dimensional module occurs with multiplicity 3, and the two-dimensional simple module occurs with multiplicity 7.

Solution to Exercise 5.48.

(i) Let V be a simple $\mathbb{C}H$-module with representation $T : H \to GL(V)$ whose character is χ. Then $T \circ \varphi : G \to GL(V)$ is a representation of G on V whose character is $\chi \circ \varphi$. Since the $\mathbb{C}G$-module V must be a direct sum of simple $\mathbb{C}G$-modules, its character $\chi \circ \varphi$ is a sum of irreducible characters of G.

(ii) Continue with the notation of the previous part. Since φ is surjective, the representing transformations $(T \circ \varphi)(g) = T(\varphi(g))$ for $g \in G$ are exactly the same (as a set of transformations) as the $T(h)$ for $h \in H$, so the invariant subspaces of V for G are the same as for H, i.e. just $\{0\}$ and V. Thus V is a simple $\mathbb{C}G$-module, so $\chi \circ \varphi \in \widehat{G}$.

(iii) Since every character of H is a sum of elements of $\widehat{H}$, it is enough to show that $\langle \chi \circ \varphi, \pi \rangle \neq 0$ for some (not necessarily irreducible) character χ of H. Take $\chi = \chi_{\mathbb{C}H}$, the character of the regular representation of H. Since $\ker \varphi = \{1\}$,

$$(\chi \circ \varphi)(g) = \begin{cases} |H|, & \text{if } g = 1, \\ 0, & \text{if } g \neq 1, \end{cases}$$

so $\langle \chi \circ \varphi, \pi \rangle = |G|^{-1}|H|\pi(1) \neq 0$.

In fact, there is a way to construct an induced character $\mathrm{Ind}_G^H(\pi)$ of H which has the property known as Frobenius reciprocity:

$$\langle \chi \circ \varphi, \pi \rangle = \langle \chi, \mathrm{Ind}_G^H(\pi) \rangle, \text{ for all } \chi \in \widehat{H}.$$

Here the left-hand side is an inner product in $\mathcal{C}(G)$ while the right-hand side is an inner product in $\mathcal{C}(H)$. Unfortunately, this course is too short to cover induced characters.

Solution to Exercise 5.49. It is clear that V^G is a $\mathbb{C}G$-submodule of V; by Maschke's Theorem, there is a complementary $\mathbb{C}G$-submodule W'. If $\{v_1, \cdots, v_m\}$ is a basis of V^G, then $V^G = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_m$ is a decomposition of V^G into $\mathbb{C}G$-submodules, each isomorphic to the one-dimensional trivial module $\mathbb{C}$. Moreover, W' can be written as a direct sum $W_1 \oplus \cdots \oplus W_k$ of simple $\mathbb{C}G$-submodules; no W_i can be isomorphic to $\mathbb{C}$, for if it were, then $W_i \subseteq V^G$, contrary to the assumption that $W' \cap V^G = \{0\}$. So $m = \dim V^G$ is exactly the multiplicity with which the simple module $\mathbb{C}$ occurs in V, and hence equals $\langle \chi_V, 1 \rangle$.

We now want to see what this means in the case where $V = \mathbb{C}X$ for X a finite set on which G acts. An element $\sum_{x \in X} a_x x$ of $\mathbb{C}X$ is G-invariant if and only if

$$\sum_{x \in X} a_x x = g(\sum_{x \in X} a_x x) = \sum_{x \in X} a_x gx \text{ for all } g \in G,$$

which is equivalent to saying that the coefficients of all the elements in any G-orbit in X are the same. So if $\mathcal{O}_1, \cdots, \mathcal{O}_s$ are the orbits of G in X, the elements $\sum_{x \in \mathcal{O}_1} x, \cdots, \sum_{x \in \mathcal{O}_s} x$ form a basis of $(\mathbb{C}X)^G$. Thus

$$\dim(\mathbb{C}X)^G = s = |G \backslash X|.$$

On the other hand, we know that $\chi_{\mathbb{C}X}(g) = |X^g|$ for all $g \in G$, so

$$\langle \chi_{\mathbb{C}X}, 1 \rangle = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

So the fact that $\dim(\mathbb{C}X)^G = \langle \chi_{\mathbb{C}X}, 1 \rangle$ is exactly Burnside's Lemma.

Solution to Exercise 5.50.

(i) It is clear that $(X \times X)^g = X^g \times X^g$. So Burnside's Lemma gives

$$|G \backslash (X \times X)| = \frac{1}{|G|} \sum_{g \in G} |X^g|^2 = \langle \chi_{\mathbb{C}X}, \chi_{\mathbb{C}X} \rangle,$$

where the second equality uses the obvious fact that $\overline{|X^g|} = |X^g|$.

(ii) Let W denote the submodule of $\mathbb{C}X$ in the question. We have a direct sum $\mathbb{C}X = \mathbb{C}\{\sum_{x \in X} x\} \oplus W$, where the first summand is clearly isomorphic to the trivial $\mathbb{C}G$-module. If $|X| \geq 3$, then $\dim W \geq 2$, so W is certainly not isomorphic to the trivial module; thus W is simple if and only if $\mathbb{C}X$ is the direct sum of two non-isomorphic simple $\mathbb{C}G$-modules. By Corollary 5.42, the latter condition is equivalent to $\langle \chi_{\mathbb{C}X}, \chi_{\mathbb{C}X} \rangle = 2$, so the result follows by the previous part.

Solution to Exercise 5.51.

(i) It is clear that the map $\overline{T}(g) : \overline{V} \to \overline{V} : \overline{v} \mapsto \overline{gv}$ is $\mathbb{C}$-linear, because it is the composition of a conjugate-linear map, a linear map, and another conjugate-linear map. So it suffices to check that $\overline{T}(gh) = \overline{T}(g)\overline{T}(h)$ (this incorporates the fact that $\overline{T}(g)$ is invertible, since clearly $\overline{T}(1)$ is the identity of $\overline{V}$). For any $\overline{v} \in \overline{V}$, we have

$$\overline{T}(gh)(\overline{v}) = \overline{ghv} = \overline{T}(g)(\overline{hv}) = \overline{T}(g)(\overline{T}(h)(\overline{v})),$$

as required.

(ii) Let $v_1, \cdots, v_n$ be any basis of V; then $\overline{v_1}, \cdots, \overline{v_n}$ is a basis of $\overline{V}$. With respect to these bases, it is clear that the matrix representing g on $\overline{V}$ is the complex conjugate of the matrix representing g on V. Taking traces, we get the result.

Solution to Exercise 5.52.

(i) The map $\langle\cdot,\cdot\rangle$ is clearly sesquilinear and conjugate-symmetric, since $\langle\cdot,\cdot\rangle_0$ has both these properties. Moreover,

$$\langle v,v\rangle = \sum_{g\in G}\langle gv,gv\rangle_\circ \geq 0,$$

with equality if and only if $gv = 0$ for all $g \in G$, which is obviously equivalent to $v = 0$. So $\langle\cdot,\cdot\rangle$ is an inner product. For all $g \in G$, $v, w \in V$, we have

$$\begin{aligned}\langle gv,gw\rangle &= \sum_{h\in G}\langle hgv,hgw\rangle_\circ\\ &= \sum_{h'\in G}\langle h'v,h'w\rangle_\circ \quad \text{(substituting } h = h'g^{-1}\text{)}\\ &= \langle v,w\rangle,\end{aligned}$$

as required. That is, all the representing transformations are unitary with respect to the inner product $\langle\cdot,\cdot\rangle$. (For any inner product $\langle\cdot,\cdot\rangle$ on a finite-dimensional vector space V, the unitary group of $\langle\cdot,\cdot\rangle$ is the subgroup of $GL(V)$ consisting of linear transformations τ such that $\langle\tau(v),\tau(w)\rangle = \langle v,w\rangle$ for all $v, w \in V$.)

(ii) The goal of this question is to explicitly construct the $\mathbb{C}G$-module isomorphism $\overline{V} \cong V^*$ whose existence is guaranteed by Exercise 5.51(ii).

It is easy to see that φ is $\mathbb{C}$-linear, since it is the composition of the conjugate-linear map $\overline{w} \mapsto w$ with the conjugate-linear map $w \mapsto \langle\cdot,w\rangle$. It is injective because if $\langle\cdot,w\rangle$ is the zero function on V, then in particular $\langle w,w\rangle = 0$ which implies that $w = 0$. Since $\overline{V}$ and V^* have the same dimension, namely $\dim V$, φ is an isomorphism of vector spaces. Finally, we must show that φ respects the action of G. For any $g \in G$, $v \in V$ and $\overline{w} \in \overline{V}$, we have

$$\begin{aligned}\varphi(g\overline{w})(v) &= \varphi(\overline{gw})(v) \quad \text{(by the definition of the } G\text{-action on } \overline{V}\text{)}\\ &= \langle v,gw\rangle \quad \text{(by the definition of } \varphi\text{)}\\ &= \langle g^{-1}v,w\rangle \quad \text{(by part (i))}\\ &= \varphi(\overline{w})(g^{-1}v) \quad \text{(by the definition of } \varphi\text{)}\\ &= (g\varphi(\overline{w}))(v) \quad \text{(by the definition of the } G\text{-action on } V^*\text{)},\end{aligned}$$

so $\varphi(g\overline{w}) = g\varphi(\overline{w})$ as required.

Solution to Exercise 5.61.

(i) The group D_4 (geometrically, the symmetry group of a square) has eight elements, which as usual we write $1, x, x^2, x^3, y, xy, x^2y, x^3y$. (Remember the relations: $x^4 = y^2 = 1$, $yxy = x^{-1}$.) To find the conjugacy classes, consider the effect of conjugating by x and y:

$$x(x^a)x^{-1} = x^a,\ x(x^a y)x^{-1} = x^{a+2}y,$$
$$y(x^a)y^{-1} = x^{-a},\ y(x^a y)y^{-1} = x^{-a}y.$$

From this it is clear that there are five conjugacy classes:

$$\{1\},\ \{x^2\},\ \{x, x^3\},\ \{y, x^2y\},\ \{xy, x^3y\}.$$

(Geometrically, these are the identity, half-turn, rotations through $\frac{\pi}{2}$ in either direction, reflections in a line joining opposite corners of the square, and reflections in a line joining midpoints of opposite sides of the square.) So we seek five irreducible characters.

As we saw in Example 5.10, there are four one-dimensional characters $\psi : D_4 \to \mathbb{C}^\times$, because each of $\psi(x)$ and $\psi(y)$ can be independently chosen to be 1 or -1. The remaining irreducible character χ must be 2-dimensional, so that the sum of squares of dimensions will be 8 as required. We can work out its other values using the orthogonality of the first column with each of the other columns (actually, the values on the classes of x, y, and xy must be zero, because χ must be unchanged when multiplied by the one-dimensional characters). The result is:

	1	x^2	x	y	xy
	1	1	2	2	2
1	1	1	1	1	1
ψ_1	1	1	1	-1	-1
ψ_2	1	1	-1	1	-1
$\psi_1\psi_2$	1	1	-1	-1	1
χ	2	-2	0	0	0

If we use the representing matrices of our geometric representation $D_4 \to GL(\mathbb{R}^2)$ to make a two-dimensional complex representation, we find that its character is exactly χ. So this is how to construct the two-dimensional simple $\mathbb{C}D_4$-module.

Now under the injective homomorphism $D_4 \to S_4$, x maps to a 4-cycle, x^2 and xy map to products of disjoint transpositions, and y maps to a 2-cycle. Thus the restrictions of the irreducible characters of S_4, as found in Example 5.57, to the group D_4 are:

	1	x^2	x	y	xy
$1\vert_{D_4}$	1	1	1	1	1
$\varepsilon\vert_{D_4}$	1	1	-1	-1	1
$\varrho\vert_{D_4}$	3	-1	-1	1	-1
$\varepsilon\varrho\vert_{D_4}$	3	-1	1	-1	-1
$\pi\vert_{D_4}$	2	2	0	0	2

It is easy to calculate (by finding inner products, if necessary) that

$$\begin{aligned} 1|_{D_4} &= 1, \\ \varepsilon|_{D_4} &= \psi_3, \\ \varrho|_{D_4} &= \psi_2 + \chi, \\ \varrho'|_{D_4} &= \psi_1 + \chi, \\ \pi|_{D_4} &= 1 + \psi_3. \end{aligned}$$

(ii) As in part (i), we can show that D_5 has four conjugacy classes:

$$\{1\},\ \{x, x^4\},\ \{x^2, x^3\},\ \{y, xy, x^2y, x^3y, x^4y\}.$$

(In terms of symmetries of the regular pentagon, these are the identity, rotations through $\frac{2\pi}{5}$ in either direction, rotations through $\frac{4\pi}{5}$ in either direction, and reflections about a line joining a vertex to the midpoint of the opposite side.) As we saw in Example 5.10, there are two one-dimensional characters $\psi : D_5 \to \mathbb{C}^\times$, because $\psi(x)$ must be 1 and $\psi(y)$ can be 1 or -1. Thus the sum of the squares of the other two dimensions is 8, and they must both be 2-dimensional. So the character table has the form:

	1	x	x^2	y
	1	2	2	5
1	1	1	1	1
ψ	1	1	1	-1
χ_1	2	a	b	c
χ_2	2	d	e	f

We know that $a, b, c, d, e, f \in \mathbb{R}$ since every element of D_5 is conjugate to its inverse. Now the third row has to be weighted orthogonal to the first two rows, which gives

$$2 + 2a + 2b + 5c = 2 + 2a + 2b - 5c = 0.$$

So $c = 0$ and $b = -1-a$; similarly $f = 0$ and $e = -1-d$. Moreover, the first two columns have to be orthogonal, which tells us that $d = -1-a$. Finally, the third and fourth rows have to be weighted orthogonal, so

$$4 + 2a(-1-a) + 2(-1-a)a = 0,$$

which implies that $a^2 + a - 1 = 0$, or $a = \frac{-1\pm\sqrt{5}}{2}$. The choice of sign is just a matter of swapping the last two rows, so the complete character table is

	1	x	x^2	y
	1	2	2	5
1	1	1	1	1
ψ	1	1	1	-1
χ_1	2	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	0
χ_2	2	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	0

If we use the representing matrices of our geometric representation $D_5 \to GL(\mathbb{R}^2)$ to make a two-dimensional complex representation, we find that its character is exactly the first of these two-dimensional irreducible characters (this uses the trigonometry facts $\cos(\frac{2\pi}{5}) = \frac{-1+\sqrt{5}}{4}$ and $\cos(\frac{4\pi}{5}) = \frac{-1-\sqrt{5}}{4}$). The other two-dimensional representation is obtained by composing this one with the group automorphism of D_5 which sends x to x^2 and preserves y.

Under the homomorphism $D_5 \to S_5$, x and x^2 map to 5-cycles, and y maps to the product of disjoint transpositions. So the image of $D_5 \to S_5$ lies in A_5, which implies that for every $\chi \in \widehat{S_5}$, the restriction of $\varepsilon\chi$ to D_5 will be the same as that of χ. With notation as in Example 5.59, the restrictions are:

	1	x	x^2	y
$1\vert_{D_5}$	1	1	1	1
$\varrho\vert_{D_5}$	4	-1	-1	0
$\nu\vert_{D_5}$	5	0	0	1
$\pi\vert_{D_5}$	6	1	1	-2

It is easy to calculate that

$$\begin{aligned}
1|_{D_5} &= 1,\\
\varrho|_{D_5} &= \chi_1 + \chi_2,\\
\nu|_{D_5} &= 1 + \chi_1 + \chi_2,\\
\pi|_{D_5} &= 2\psi + \chi_1 + \chi_2.
\end{aligned}$$

Since $\pi|_{A_5}$ already splits as $\pi_1 + \pi_2$, it is slightly more informative to restrict the characters π_1 and π_2 of A_5 to D_5:

$$\pi_1|_{D_5} = \psi + \chi_1, \ \ \pi_2|_{D_5} = \psi + \chi_2,$$

which 'explains' why $\sqrt{5}$ appears in the character table of both D_5 and A_5.

Solution to Exercise 5.62. It is easy to work out that the conjugacy classes in Q are

$$\{1\}, \ \{-1\}, \ \{i, -i\}, \ \{j, -j\}, \ \{k, -k\}.$$

So we seek five irreducible characters. First we find the one-dimensional characters $\psi : Q \to \mathbb{C}^\times$: the fact that $\psi(i) = \psi(-i)$ implies that $\psi(-1) = 1$, so each of $\psi(i)$ and $\psi(j)$ must be either 1 or -1. All four possibilities give a valid character, so there is only one irreducible character χ left to find, which must be 2-dimensional since $|Q| = 8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$. Finding its values by column orthogonality, we obtain the character table:

	1	-1	$\pm i$	$\pm j$	$\pm k$
1	1	1	1	1	1
ψ_1	1	1	1	-1	-1
ψ_2	1	1	-1	1	-1
$\psi_1\psi_2$	1	1	-1	-1	1
χ	2	-2	0	0	0

We saw an irreducible matrix representation $Q \to GL_2(\mathbb{C})$ with character χ in Exercise 4.91(iv).

Note that the character table of Q is the same as the character table of D_4 found in Exercise 5.61(i), despite the fact that $Q \ncong D_4$ (see Exercise 4.91(i)). So it is possible for non-isomorphic groups to have the same character table.

Solution to Exercise 5.63. Let $\tau : \mathcal{C}(G) \to \mathcal{C}(G)$ be the $\mathbb{C}$-linear map defined by $(\tau f)(g) = f(g^{-1})$. In terms of the characteristic-function basis $\{f_1, \cdots, f_s\}$ of $\mathcal{C}(G)$, τ preserves those f_i for which $c_i^{-1} = c_i$, and swaps f_i and f_j if $c_i^{-1} = c_j$. Hence the fixed-point subspace $\mathcal{C}(G)^\tau$ is spanned by

$$\{f_i \,|\, c_i^{-1} = c_i\} \cup \{f_i + f_j \,|\, c_i^{-1} = c_j\},$$

and therefore $\dim \mathcal{C}(G)^\tau = t + \frac{s-t}{2} = \frac{s+t}{2}$. But we also know that $\tau(\chi_i) = \overline{\chi_i}$ for every $\chi_i \in \widehat{G}$. So by the same reasoning, $\dim \mathcal{C}(G)^\tau = \frac{s+u}{2}$, where u is the number of real-valued χ_i's. Hence $u = t$ as claimed.

Solution to Exercise 5.64.

(i) As in the proof of Theorem 5.36, we know that $\tilde{e}_{\chi_i}$ acts on V_j as a $\mathbb{C}G$-module endomorphism, which must be some scalar multiplication by Schur's Lemma. The trace of this scalar multiplication is $\frac{1}{|G|}\sum_{g\in G}\chi_i(g^{-1})\chi_j(g) = \langle \chi_i, \chi_j \rangle$, which by Schur's Orthogonality Relations is 1 if $i = j$ and 0 otherwise. Hence the scalar is $\frac{1}{\dim V_i} = \frac{1}{\chi_i(1)}$ if $i = j$ and 0 otherwise, as claimed.

(ii) We may assume that $W_1, \cdots, W_s$ are the simple summands whose character is χ_i. Any $w \in W$ can be written uniquely as $w_1 + \cdots + w_k$ where $w_i \in W_i$. By the previous part,

$$\tilde{e}_{\chi_i}(w_1 + \cdots + w_k) = \frac{1}{\chi_i(1)}(w_1 + \cdots + w_s).$$

So the sum $W_1 \oplus \cdots \oplus W_s$ can be described alternatively as $\tilde{e}_{\chi_i}W$, which is independent of the chosen direct sum decomposition.

As observed in Remark 4.101, the simple summands $W_1, \cdots, W_s$ individually are generally not uniquely determined. This is reminiscent of the way that primary components of a module over a PID are uniquely defined, although their indecomposable summands are not (see Example 2.122).

(iii) In the previous part we saw that for any $w \in W$, $\tilde{e}_{\chi_i}w$ lies in the sum of those simple summands which have character χ_i, so $\tilde{e}^2_{\chi_i}w = \frac{1}{\chi_i(1)}\tilde{e}_{\chi_i}w$ and $\tilde{e}_{\chi_j}\tilde{e}_{\chi_i}w = 0$ for all $j \neq i$. In particular, this applies when $W = \mathbb{C}G$ and $w = 1$, which gives the result.

(iv) Let $g \in G$ and let ζ be a $|G|$th root of 1. We have

$$\begin{aligned}\frac{|G|}{\chi_i(1)}\zeta g\tilde{e}_{\chi_i} &= |G|\zeta g\tilde{e}^2_{\chi_i} \quad \text{(by part (iii))}\\ &= \zeta g\,(\sum_{h\in G}\chi_i(h^{-1})h)\tilde{e}_{\chi_i} \quad \text{(by the definition of } \tilde{e}_{\chi_i})\\ &= \sum_{h\in G}\zeta\chi_i(h^{-1})gh\tilde{e}_{\chi_i}.\end{aligned}$$

Now $\chi_i(h^{-1})$ is the sum of the eigenvalues of a matrix representing h^{-1}, and the order of h^{-1} in the group divides $|G|$, so these eigenvalues are $|G|$th roots of 1, and our expression is of the right form to be contained in M_{χ_i}, as required.

(v) Since M_{χ_i} is a finitely-generated $\mathbb{Z}$-module, and torsion-free because it is contained in a complex vector space, it must have some basis $m_1, \cdots, m_t$ over $\mathbb{Z}$. By the previous part, $\frac{|G|}{\chi_i(1)}m_1 = \sum_{i=1}^{t} a_i m_i$ for some $a_i \in \mathbb{Z}$, which implies that $|G|m_1 = \sum_{i=1}^{t} \chi_i(1) a_i m_i$. Comparing coefficients of m_1, we see that $|G| = \chi_i(1)a_1$, which shows that $\chi_i(1)$ divides $|G|$.