

Solutions to Chapter 6

Solution to Exercise 6.30. The character table of $D_3 \cong S_3$ (given in Example 5.55) is all real. The one-dimensional $\mathbb{C}D_3$ -modules are thus automatically defined over \mathbb{R} . The two-dimensional simple $\mathbb{C}D_3$ -module with character ϱ is also defined over \mathbb{R} : indeed, it can be obtained by complexifying our original faithful representation of D_3 on \mathbb{R}^2 . Hence there are three simple $\mathbb{R}D_3$ -modules, which it is natural to call \mathbb{R} , \mathbb{R}_ε and $(\mathbb{R}^2)_\varrho$. It follows that there are four two-dimensional $\mathbb{R}D_3$ -modules up to isomorphism, namely $\mathbb{R} \oplus \mathbb{R}$, $\mathbb{R}_\varepsilon \oplus \mathbb{R}_\varepsilon$, $\mathbb{R} \oplus \mathbb{R}_\varepsilon$, and $(\mathbb{R}^2)_\varrho$. These correspond to the four equivalence classes of representations of D_3 on \mathbb{R}^2 seen in Exercise 4.39.

Similarly, the character table of D_4 (found in Exercise 5.61(i)) is all real, and the two-dimensional simple $\mathbb{C}D_4$ -module is the complexification of our original faithful representation of D_4 on \mathbb{R}^2 . Hence there are five simple $\mathbb{R}D_4$ -modules, which it is natural to call \mathbb{R} , \mathbb{R}_{ψ_1} , \mathbb{R}_{ψ_2} , $\mathbb{R}_{\psi_1\psi_2}$, and $(\mathbb{R}^2)_\chi$. It follows that there are eleven two-dimensional $\mathbb{R}D_4$ -modules up to isomorphism: ten ($= 4 + \binom{4}{2}$) formed by taking the direct sum of two one-dimensional modules, and the simple two-dimensional module $(\mathbb{R}^2)_\chi$. Hence there are eleven equivalence classes of representations of D_4 on \mathbb{R}^2 . Incidentally, note that only the last equivalence class (containing our original representation) consists of faithful representations; this is clear from the fact that x^2 acts trivially on all the one-dimensional modules.

The character table of D_5 is also all real, as seen in Exercise 5.61(ii); moreover, the two-dimensional simple $\mathbb{C}D_5$ -modules can be obtained by complexifying real representations. So there are four simple $\mathbb{R}D_5$ -modules, which it is natural to call \mathbb{R} , \mathbb{R}_ψ , $(\mathbb{R}^2)_{\chi_1}$, and $(\mathbb{R}^2)_{\chi_2}$. It follows that there are five two-dimensional $\mathbb{R}D_5$ -modules up to isomorphism: three obtained by tak-

ing the direct sum of two one-dimensional modules, and the two simple two-dimensional modules. Hence there are five equivalence classes of representations of D_5 on \mathbb{R}^2 . Since x acts trivially on both one-dimensional modules, only the simple two-dimensional modules afford faithful representations.

Solution to Exercise 6.31.

(i) It is geometrically clear that G acts transitively on the set of vertices, and the stabilizer of each vertex is a cyclic group of order 5. By the orbit-stabilizer relation, $|G| = 12 \times 5 = 60$. Explicitly, the 60 rotations are:

- the identity;
- one 180° rotation about each of the fifteen edge axes (lines joining the midpoints of opposite edges);
- two 120° rotations about each of the ten face axes (lines joining the centres of opposite faces);
- two 72° rotations about each of the six vertex axes (lines joining opposite vertices); and
- two 144° rotations about each of the six vertex axes.

As you would expect, these are the five conjugacy classes in G .

- (ii) It is geometrically clear that H acts transitively on this set of six edge-midpoints, and the stabilizer of each edge-midpoint consists of the identity and the corresponding 180° rotation, so by the orbit-stabilizer relation, $|H| = 6 \times 2 = 12$.
- (iii) Let $X = G/H$, the set of cosets of H in G . Clearly G acts on X by left multiplication, so we get a group homomorphism $\varphi : G \rightarrow \text{Sym}(X) \cong S_5$. The kernel of φ is the set

$$\{g \in G \mid gg'H = g'H, \forall g' \in G\} = \{g \in G \mid (g')^{-1}gg' \in H, \forall g' \in G\},$$

i.e. the union of those conjugacy classes which are completely contained in H ; from the sizes found in part (i) it is clear that the identity is the only such class, so φ is injective. Thus φ gives an isomorphism between G and a subgroup of order 60 inside S_5 which has order 120;

this subgroup must be A_5 , since a subgroup of index 2 is automatically the kernel of a homomorphism to $\{\pm 1\}$, and the only non-trivial homomorphism $S_5 \rightarrow \{\pm 1\}$ is the sign character. Hence $G \cong A_5$ (with the conjugacy classes of G found in (i) corresponding to the conjugacy classes of A_5 found in Example 5.60).

- (iv) Composing the isomorphism $A_5 \cong G$ with the obvious inclusion $G \hookrightarrow GL(\mathbb{R}^3)$, we get a representation of A_5 on \mathbb{R}^3 , or in other words a three-dimensional $\mathbb{R}A_5$ -module. If this were not simple it would have to have a one-dimensional submodule, which would mean that all the rotations in G had a common eigenvector. But the only eigenvectors of the rotations of order 5 are the vectors along the axis of rotation, so this is clearly not true. Thus we have defined a simple three-dimensional $\mathbb{R}A_5$ -module, whose character must be one of the last two rows of the character table of A_5 (see Example 5.60). Which of the two it is depends on the identification $\text{Sym}(X) \cong S_5$ used above, i.e. on an arbitrary ordering of the cosets of H in G ; changing this by an odd permutation will change one character to the other.

Solution to Exercise 6.32.

- (i) By Corollary 5.35, we have $\dim_{\mathbb{C}} \text{End}_{\mathbb{C}G}(U_{\mathbb{C}}) = \langle \chi_{U_{\mathbb{C}}}, \chi_{U_{\mathbb{C}}} \rangle = \langle \chi_U, \chi_U \rangle$, so it suffices to show that $\dim_{\mathbb{R}} \text{End}_{\mathbb{R}G}(U) = \dim_{\mathbb{C}} \text{End}_{\mathbb{C}G}(U_{\mathbb{C}})$. If we choose a basis v_1, \dots, v_n of U , then \mathbb{R} -linear transformations of U correspond to $n \times n$ real matrices, and \mathbb{C} -linear transformations of $U_{\mathbb{C}}$ correspond to $n \times n$ complex matrices. The condition that the linear transformation is an endomorphism over the group algebra is equivalent, in each case, to the condition that the matrix commutes with all the representing matrices. This is some system of linear equations in n^2 unknowns (the entries of the matrix), with real coefficients which are the same in both cases. By basic linear algebra, the dimension of the space of solutions is the same whether we solve the system in the real numbers or in the complex numbers.
- (ii) Combining part (i) with Theorem 6.16, we see that $A = \text{End}_{\mathbb{R}G}(U)$ is either 1-dimensional, 2-dimensional, or 4-dimensional over \mathbb{R} . We also know that A is an \mathbb{R} -algebra (so it contains an isomorphic copy of \mathbb{R} , the scalar multiples of the identity element 1), and a division

ring by Schur's Lemma. We can actually prove that \mathbb{R} , \mathbb{C} , and \mathbb{H} are (up to isomorphism) the only \mathbb{R} -algebras of these dimensions which are division rings.

If A is 1-dimensional, it must be equal to the copy of \mathbb{R} it contains, so that case is easy. If A is 2-dimensional, it must have a basis $\{1, a\}$, and since the basis elements commute it must be commutative and hence a field. But basic field theory shows that there is a unique degree-2 field extension of \mathbb{R} , namely \mathbb{C} .

Henceforth we assume that A is 4-dimensional, and aim to prove that $A \cong \mathbb{H}$. Choose an element $a \in A \setminus \mathbb{R}$. The minimal polynomial of a over \mathbb{R} is an irreducible monic element of $\mathbb{R}[x]$ which is not of the form $x - b$ for $b \in \mathbb{R}$, so it must be $x^2 + px + q$ for some $p, q \in \mathbb{R}$ such that $p^2 - 4q < 0$. Hence the \mathbb{R} -span of 1 and a is a subring of A isomorphic to \mathbb{C} ; we may as well identify it with \mathbb{C} . So we can assume that A contains \mathbb{C} as a subring. If $i \in \mathbb{C}$ commuted with anything in $A \setminus \mathbb{C}$, then A , being two-dimensional over \mathbb{C} , would be commutative and hence a degree-2 field extension of \mathbb{C} , which is impossible. So the order-2 \mathbb{R} -linear transformation $A \rightarrow A : a \mapsto iai^{-1}$ has 1-eigenspace \mathbb{C} and a two-dimensional (-1) -eigenspace, say B . For any $b \in B$, $ib = -bi$, so $ib^2 = b^2i$, which means that $b^2 \in \mathbb{C}$; but also b^2 commutes with b , which forces $b^2 \in \mathbb{R}$. Since 1 and b span a subring isomorphic to \mathbb{C} , we have $b^2 < 0$, so the element $j = b/\sqrt{-b^2}$ in B satisfies $j^2 = -1$. Set $k = ij \in B$, which satisfies $k^2 = ijij = -i^2j^2 = -1$, and is clearly not a real scalar multiple of j . We have found a basis $\{1, i, j, k\}$ of A which satisfies all the relations of the usual quaternion basis, so $A \cong \mathbb{H}$.

Recall that we explicitly computed an endomorphism algebra of each of these three types in Exercise 4.106.

Solution to Exercise 6.45. The function $\vartheta : G \rightarrow \mathbb{C}$ which takes g to the number of solutions of $h^2 = g$ is certainly a class function, since $(xhx^{-1})^2 = xh^2x^{-1}$. We have

$$\langle \vartheta, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \vartheta(g) \chi_i(g^{-1}) = \frac{1}{|G|} \sum_{h \in G} \chi_i(h^{-2}),$$

which equals the Frobenius-Schur indicator $FS(\chi_i)$ by definition. Hence $\vartheta = \sum_{i=1}^s FS(\chi_i) \chi_i$, as required.

An interesting special case of this result is that the number of solutions of the equation $h^2 = 1$ in G is $\sum_{i=1}^s FS(\chi_i) \dim V_i$. So the number of involutions (i.e. elements of order 2) in G is $\sum_{i=2}^s FS(\chi_i) \dim V_i$ (removing 1 for the identity), which is bounded above by $\sum_{i=2}^s \dim V_i$. Since $\sum_{i=2}^s (\dim V_i)^2 = |G| - 1$, the Cauchy-Schwarz inequality implies that $\sum_{i=2}^s \dim V_i$ is in turn bounded above by $\sqrt{(s-1)(|G|-1)}$. This upper bound on the number of involutions is a purely group-theoretic fact, but would be very difficult to prove without character theory.

Solution to Exercise 6.46.

- (i) It suffices to show that this map is injective. Recall that in a finite group G , every element g satisfies $g^{|G|} = 1$. If $|G|$ is odd, then we have $g = g^{|G|+1} = (g^2)^u$ where $u = \frac{|G|+1}{2}$. So $g^2 = h^2$ implies that $(g^2)^u = (h^2)^u$, which says that $g = h$ as required.
- (ii) For any irreducible character χ_i of G , we have

$$FS(\chi_i) = \frac{1}{|G|} \sum_{g \in G} \chi_i(g^2) = \frac{1}{|G|} \sum_{g' \in G} \chi_i(g'),$$

by the previous part. But the last expression is by definition $\langle \chi_i, 1 \rangle$, which is 1 if χ_i is the trivial character and 0 otherwise. So by Theorem 6.41, the only real-valued χ_i is the trivial character.

- (iii) By Exercise 5.63, it suffices to show that the only self-inverse conjugacy class of G is the class $\{1\}$. Suppose that c is a self-inverse conjugacy class; then we have a map $\nu : c \rightarrow c : g \mapsto g^{-1}$ such that ν^2 is the identity. Since $|c|$ divides $|G|$, $|c|$ must be odd, so ν must have a fixed point. So there is some $g \in c$ such that $g = g^{-1}$, i.e. $g^2 = 1$. By part (i), this implies that $g = 1$ as required.

Solution to Exercise 6.47.

- (i) Any element of Q_3 can be written as a ‘word’ in the ‘letters’ x and y (since $x^{-1} = x^5$ and $y^{-1} = y^3$, we don’t need to consider the inverses as separate letters). The last relation in the presentation can be rewritten

$yx = x^5y$, and this can be used repeatedly to push every y in the word to the right of all x 's. Hence every element of Q_3 can be written in the form x^ay^b , where we can assume that $0 \leq a \leq 5$ because $x^6 = 1$ and $0 \leq b \leq 1$ because $y^2 = x^3$. So Q_3 has at most twelve elements, namely

$$1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y.$$

We need to say "at most" because it is theoretically possible that the relations might also force some equalities between these twelve words.

To check that this doesn't happen, we need an explicit realization of the group, and the easiest to construct uses permutations. Number the words in the above list 1 to 12 in the above order. The permutation induced by left multiplying by x (and using the relation $x^6 = 1$) is $(1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12)$. The permutation induced by left multiplying by y (and using the relations) is $(1\ 7\ 4\ 10)(2\ 12\ 5\ 9)(3\ 11\ 6\ 8)$. This suggests that we would obtain a group homomorphism $\Phi : Q_3 \rightarrow S_{12}$ by sending x and y to these permutations, which is easily verified. Some more easy calculations show that the twelve words map to twelve different permutations, confirming that Q_3 has 12 elements.

To find the conjugacy classes, we consider the effect of conjugating by x and y :

$$\begin{aligned} x(x^a)x^{-1} &= x^a, & x(x^ay)x^{-1} &= x^{a+2}y, \\ y(x^a)y^{-1} &= x^{-a}, & y(x^ay)y^{-1} &= x^{-a}y. \end{aligned}$$

From this it is clear that there are six conjugacy classes:

$$\{1\}, \{x^3\}, \{x, x^5\}, \{x^2, x^4\}, \{y, x^2y, x^4y\}, \{xy, x^3y, x^5y\}.$$

- (ii) We first find the one-dimensional characters, i.e. group homomorphisms $\psi : Q_3 \rightarrow \mathbb{C}^\times$. The relation $\psi(y)\psi(x)\psi(y)^{-1} = \psi(x)^{-1}$ implies that $\psi(x) = 1$ or -1 . In the first case, $\psi(x)^3 = \psi(y)^2$ shows that $\psi(y) = \pm 1$; in the second case, it shows that $\psi(y) = \pm i$. Each of these four possibilities gives a one-dimensional character: these are in fact the powers $1, \psi, \psi^2, \psi^3$ where $\psi(y) = i$. (That is, the group Q_3^\vee is isomorphic to C_4 .) The remaining two simple $\mathbb{C}Q_3$ -modules must both be two-dimensional, because the sum of the squares of the dimensions is $|Q_3| = 12$.

Now note that there is a surjective homomorphism $\varphi : Q_3 \rightarrow D_3$ which sends x to x and y to y . So one of the two-dimensional irreducible

characters must be the composition $\varrho \circ \varphi$ where ϱ is the two-dimensional irreducible character of D_3 . Since $\psi(\varrho \circ \varphi)$ is not the same as $\varrho \circ \varphi$, it must be the remaining irreducible character. The character table is:

	1	x^3	x	x^2	y	xy
	1	1	2	2	3	3
1	1	1	1	1	1	1
ψ	1	-1	-1	1	i	$-i$
ψ^2	1	1	1	1	-1	-1
ψ^3	1	-1	-1	1	$-i$	i
$\varrho \circ \varphi$	2	2	-1	-1	0	0
$\psi(\varrho \circ \varphi)$	2	-2	1	-1	0	0

There are many alternative ways to find the last two rows without using $\varphi : Q_3 \rightarrow D_3$.

- (iii) A one-dimensional simple $\mathbb{C}Q_3$ -module is defined over \mathbb{R} if and only if its character is real-valued, which is true for 1 and ψ^2 but not for ψ or ψ^3 . For the two-dimensional simple $\mathbb{C}Q_3$ -modules, we compute Frobenius-Schur indicators:

$$FS(\varrho \circ \varphi) = \frac{1}{12}(2 \times 2 + 4 \times (-1) + 6 \times 2) = 1,$$

$$FS(\psi(\varrho \circ \varphi)) = \frac{1}{12}(2 \times 2 + 4 \times (-1) + 6 \times (-2)) = -1.$$

So the simple module with character $\varrho \circ \varphi$ is defined over \mathbb{R} , and that with character $\psi(\varrho \circ \varphi)$ is not.

- (iv) From the previous part, it follows that there are five simple finite-dimensional $\mathbb{R}Q_3$ -modules, which it is natural to call \mathbb{R} , \mathbb{R}_{ψ^2} , $(\mathbb{R}^2)_{\psi+\psi^3}$, $(\mathbb{R}^2)_{\varrho \circ \varphi}$, and U , where U is the four-dimensional simple $\mathbb{R}Q_3$ -module with character $2\psi(\varrho \circ \varphi)$. The endomorphism ring $\text{End}_{\mathbb{R}Q_3}(U)$ is isomorphic to \mathbb{H} by Exercise 6.32(ii), so U is in fact isomorphic to \mathbb{H} with quaternions acting by right multiplication. By an argument which will be explained in Section 6.3 in the context of a different group, the fact that the representation of Q_3 on U is faithful (as may be seen from the character values) implies that Q_3 is isomorphic to a subgroup of the unit quaternions. One isomorphism is $x \mapsto \frac{1-i+j+k}{2}$, $y \mapsto \frac{i+j}{\sqrt{2}}$.

Solution to Exercise 6.49. Write the elements of \mathbb{Z}_5 as $\{0, 1, -1, 2, -2\}$. The group $GL_2(\mathbb{Z}_5)$ has 480 elements (24 choices for the first column, then 20

for the second), and $G = SL_2(\mathbb{Z}_5)$ is the kernel of the surjective determinant homomorphism $\det : GL_2(\mathbb{Z}_5) \rightarrow \mathbb{Z}_5^\times$, so $|G| = 120$. Using our canonical form theorems and some centralizer calculations, we can see that G has nine conjugacy classes:

- (1) The conjugacy class of the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which has size 1.
- (2) The conjugacy class of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which has size 1.
- (3) The conjugacy class of $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, which has size 12.
- (4) The conjugacy class of $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, which has size 12.
- (5) The conjugacy class of $\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$, which has size 12.
- (6) The conjugacy class of $\begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$, which has size 12.
- (7) The conjugacy class of $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, which has size 30.
- (8) The conjugacy class of $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, which has size 20.
- (9) The conjugacy class of $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, which has size 20.

All conjugacy classes are self-inverse, so the character table will be all real.

We now need to find a 2-to-1 group homomorphism $\varphi : G \rightarrow A_5$. Imitating the style of proof of Exercise 6.31(iii), we first need a subgroup of index 5 in G : it is straightforward to check that the subgroup H generated by $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 2 & -2 \end{pmatrix}$ has 24 elements, as required. The action of G on the set of cosets gives us a group homomorphism $\varphi : G \rightarrow S_5$, whose kernel is the union of the conjugacy classes of G which are contained in H , namely $\{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$. Since $\text{im}(\varphi)$ is an index-2 subgroup of S_5 , it must be A_5 , and we have our desired 2-to-1 group homomorphism $\varphi : G \rightarrow A_5$. Considering the orders of elements, it is easy to see that the third and fifth conjugacy classes of G map onto one of the 12-element classes of A_5 , the fourth and sixth conjugacy classes of G map onto the other 12-element class, the seventh conjugacy class maps onto the 15-element class of A_5 , and the eighth and ninth conjugacy classes of G map onto the 20-element class of A_5 .

Pulling back the irreducible characters of A_5 to G , we get five irreducible characters of G , whose dimensions are 1, 4, 5, 3, 3, their squares making a

total of 60. There are four remaining irreducible characters, whose dimensions squared must add up to $120 - 60 = 60$. It is easy to check that there are only two ways to write 60 as a sum of four squares:

$$60 = 6^2 + 4^2 + 2^2 + 2^2 = 5^2 + 5^2 + 3^2 + 1^2.$$

But there cannot be a one-dimensional character ψ among the four characters we are seeking, because the fact that it didn't arise via A_5 would imply that $\psi\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = -1$, so $\psi\left(\begin{smallmatrix} 2 & 0 \\ 0 & -2 \end{smallmatrix}\right)$ would have to be a square root of -1 and would also have to be real, as seen above. Hence the dimensions of the remaining characters are 6, 4, 2, 2. Call them $\chi_1, \chi_2, \chi_3, \chi$, where χ is one of the two-dimensional characters, and let $M : G \rightarrow GL_2(\mathbb{C})$ be a matrix representation with character χ . Schur's Lemma shows that $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts as multiplication by -1 on the $\mathbb{C}G$ -modules with these four characters. In particular, $M\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. We can now see that M must be faithful, because there is no nontrivial normal subgroup of G which does not contain $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. (A normal subgroup must be a union of conjugacy classes, and all elements but those in the first, third, and fourth conjugacy classes have $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ as one of their powers, so this is easy to check.)

Now the characters which factor through A_5 all give simple $\mathbb{C}G$ -modules which are defined over \mathbb{R} , since all the simple $\mathbb{C}A_5$ -modules are defined over \mathbb{R} . So the $g = 1$ case of Exercise 6.45 says that

$$1 + 4 + 5 + 3 + 3 + 6FS(\chi_1) + 4FS(\chi_2) + 2FS(\chi_3) + 2FS(\chi) = 2.$$

Hence we must have $FS(\chi_1) = FS(\chi_2) = FS(\chi_3) = FS(\chi) = -1$.

Exactly the same argument as in Section 6.3, using the character χ , now shows that there is an injective group homomorphism $\vartheta : G \rightarrow U(\mathbb{H})$ such that $\vartheta\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = -1$. Composing this with the 2-to-1 map $\eta : U(\mathbb{H}) \rightarrow SO_3(\mathbb{R})$ gives a group homomorphism $\eta \circ \vartheta : G \rightarrow SO_3(\mathbb{R})$ whose kernel is $\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$; this must be of the form $T' \circ \varphi$ for some injective group homomorphism $T' : A_5 \rightarrow SO_3(\mathbb{R})$.

We can regard T' as a faithful representation of A_5 on \mathbb{R}^3 ; consulting the character table of A_5 , we see that T' must give rise to one of the two simple $\mathbb{R}A_5$ -module structures on \mathbb{R}^3 . So T' is equivalent to one of the two representations of A_5 constructed in Exercise 6.31. By the same argument as in Section 6.3, we can assume that T' actually is one of these two representations, which means that the image of T' consists of the group of rotations

of our chosen icosahedron. Hence $\vartheta(G)$ is the binary icosahedral group, as claimed.

Solution to Exercise 6.50. As noted in Section 6.3, $|GL_2(\mathbb{Z}_3)| = 48$. Our canonical form theorems and some easy calculations show that $GL_2(\mathbb{Z}_3)$ has eight conjugacy classes:

- (1) The conjugacy class of the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which has size 1.
- (2) The conjugacy class of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which has size 1.
- (3) The conjugacy class of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which has size 8.
- (4) The conjugacy class of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which has size 8.
- (5) The conjugacy class of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which has size 6.
- (6) The conjugacy class of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which has size 12.
- (7) The conjugacy class of $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, which has size 6.
- (8) The conjugacy class of $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, which has size 6.

The first five classes comprise $SL_2(\mathbb{Z}_3)$, and the other elements have determinant -1 . The first six classes are self-inverse, and the other two are inverse to each other. So there must be some non-real values in the last two columns of the character table.

As seen in Section 6.3, we have a surjective group homomorphism $\varphi : GL_2(\mathbb{Z}_3) \rightarrow S_4$. We can pull back the five irreducible characters of S_4 to give five irreducible characters of $GL_2(\mathbb{Z}_3)$, again called $1, \varepsilon, \rho, \varepsilon\rho, \pi$. The other three irreducible characters must have dimensions whose squares add up to 24; the only possibility is two 2-dimensional characters and a 4-dimensional character. Since the character table must contain some non-real values, the two 2-dimensional characters form a conjugate pair $\{\tau \neq \bar{\tau}\}$, while the 4-dimensional character ω is real-valued.

As the only 4-dimensional irreducible character, ω must equal $\varepsilon\omega$, so $\omega(g) = 0$ for all g such that $g \notin \ker(\varepsilon) = SL_2(\mathbb{Z}_3)$. Hence for any $\chi' \in \widehat{GL_2(\mathbb{Z}_3)}$,

$$\langle \chi', \omega \rangle = \frac{1}{48} \sum_{g \in SL_2(\mathbb{Z}_3)} \chi'(g)\omega(g) = \frac{1}{2} \langle \chi'|_{SL_2(\mathbb{Z}_3)}, \omega|_{SL_2(\mathbb{Z}_3)} \rangle.$$

So the restriction $\omega|_{SL_2(\mathbb{Z}_3)}$ must be orthogonal in $\mathcal{C}(SL_2(\mathbb{Z}_3))$ to the characters $1, \psi, \psi^2, \varrho$ found in Section 6.3; also its self-inner product is 2. Consulting the character table of $SL_2(\mathbb{Z}_3)$, we see that $\omega|_{SL_2(\mathbb{Z}_3)}$ must be $\psi\chi + \psi^2\chi$. This determines ω completely.

By Exercise 5.48(iii), there must be some irreducible character of $GL_2(\mathbb{Z}_3)$ whose restriction to $SL_2(\mathbb{Z}_3)$ is not orthogonal to the character χ . The only remaining possibility is that τ (or equivalently $\bar{\tau}$) has this property; hence in fact $\tau|_{SL_2(\mathbb{Z}_3)} = \chi$. Now the non-real values of τ are taken on elements g with $\varepsilon(g) = -1$, and therefore it is impossible that $\varepsilon\tau = \tau$. Hence $\varepsilon\tau = \bar{\tau}$, which shows that τ takes purely imaginary values on all the classes not contained in $SL_2(\mathbb{Z}_3)$. Recalling that $\overline{\tau(g)} = \tau(g^{-1})$, we see that τ has the form

	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$
τ	1	1	8	8	6	12	6	6
	2	-2	-1	1	0	0	ia	$-ia$

for some $a \in \mathbb{R}$. The fact that $\langle \tau, \tau \rangle = 1$ means that $4+4+8+8+6a^2+6a^2 = 48$, so $a = \pm\sqrt{2}$. The choice of sign amounts to swapping the names of τ and $\bar{\tau}$, so we can assume that $a = \sqrt{2}$, and the character table is complete:

	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$
	1	1	8	8	6	12	6	6
1	1	1	1	1	1	1	1	1
ε	1	1	1	1	1	-1	-1	-1
ϱ	3	3	0	0	-1	1	-1	-1
$\varepsilon\varrho$	3	3	0	0	-1	-1	1	1
π	2	2	-1	-1	2	0	0	0
ω	4	-4	1	-1	0	0	0	0
τ	2	-2	-1	1	0	0	$\sqrt{2}i$	$-\sqrt{2}i$
$\bar{\tau}$	2	-2	-1	1	0	0	$-\sqrt{2}i$	$\sqrt{2}i$

An easy calculation shows that $FS(\omega) = 1$. So all the simple $\mathbb{C}GL_2(\mathbb{Z}_3)$ -modules with real-valued characters are defined over \mathbb{R} .

On the other hand, the binary cubic group (using the same cube as in Exercise 4.60) consists of $\pm 1, \pm i, \pm j, \pm k$, the sixteen unit quaternions of the form $\frac{\pm 1 \pm i \pm j \pm k}{2}$, and the following twenty-four unit quaternions:

$$\frac{\pm 1 \pm i}{\sqrt{2}}, \frac{\pm 1 \pm j}{\sqrt{2}}, \frac{\pm 1 \pm k}{\sqrt{2}}, \frac{\pm i \pm j}{\sqrt{2}}, \frac{\pm i \pm k}{\sqrt{2}}, \frac{\pm j \pm k}{\sqrt{2}}.$$

These 48 elements are also divided into eight conjugacy classes:

- (1) The conjugacy class of 1, which has size 1.
- (2) The conjugacy class of -1 , which has size 1.
- (3) The conjugacy class of $\frac{-1-i-j-k}{2}$, which has size 8.
- (4) The conjugacy class of $\frac{1+i+j+k}{2}$, which has size 8.
- (5) The conjugacy class of i , which has size 6.
- (6) The conjugacy class of $\frac{i+j}{\sqrt{2}}$, which has size 12.
- (7) The conjugacy class of $\frac{1+i}{\sqrt{2}}$, which has size 6.
- (8) The conjugacy class of $\frac{-1-i}{\sqrt{2}}$, which has size 6.

Note that the first five classes comprise the binary tetrahedral group, which as seen in Section 6.3 is isomorphic to $SL_2(\mathbb{Z}_3)$. The three remaining classes are all self-inverse, which shows that the character table has all real values; this proves that the binary cubic group is not isomorphic to $GL_2(\mathbb{Z}_3)$.

The character table of the binary cubic group may be found in exactly the same way as the character table of $GL_2(\mathbb{Z}_3)$, making use of the 2-to-1 projection onto S_4 and the subgroup isomorphic to $SL_2(\mathbb{Z}_3)$. The result is:

	1	-1	$\frac{-1-i-j-k}{2}$	$\frac{1+i+j+k}{2}$	i	$\frac{i+j}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$
	1	1	8	8	6	12	6	6
1	1	1	1	1	1	1	1	1
ε	1	1	1	1	1	-1	-1	-1
ϱ	3	3	0	0	-1	1	-1	-1
$\varepsilon\varrho$	3	3	0	0	-1	-1	1	1
π	2	2	-1	-1	2	0	0	0
ω	4	-4	1	-1	0	0	0	0
τ	2	-2	-1	1	0	0	$\sqrt{2}$	$-\sqrt{2}$
$\varepsilon\tau$	2	-2	-1	1	0	0	$-\sqrt{2}$	$\sqrt{2}$

This time, we have $FS(\omega) = 1$ but $FS(\tau) = FS(\varepsilon\tau) = -1$. So the two-dimensional modules with characters τ and $\varepsilon\tau$ are not defined over \mathbb{R} . There had to be such quaternion-type characters in the character table, because the group is a subgroup of $U(\mathbb{H})$.