MATH3968 – Lecture 12

Orientation

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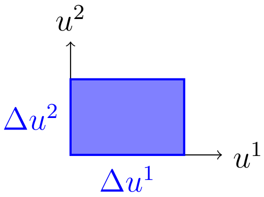
Let

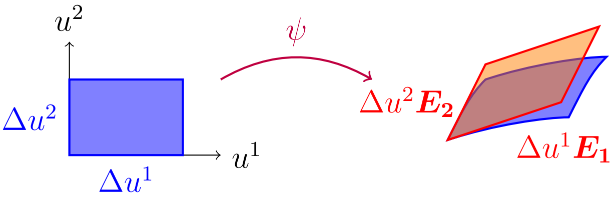
$$\begin{aligned}\phi : U &\to \Sigma \\ (u^1, u^2) &\mapsto \phi(u^1, u^2)\end{aligned}$$

be a local parameterisation near $p \in \Sigma$.

The image in Σ of the rectangle



is approximately a parallelogram with sides $\Delta u^1 \boldsymbol{E_1}$ and $\Delta u^2 \boldsymbol{E_2}$, where $\boldsymbol{E_1} = \phi_{u^1}$ and $\boldsymbol{E_2} = \phi_{u^2}$.



The parallelogram in $T_p\Sigma$ with sides $\Delta u^1 \boldsymbol{E_1}$ and $\Delta u^2 \boldsymbol{E_2}$ has area

$$|\boldsymbol{E_1} \times \boldsymbol{E_2}| \Delta u^1 \Delta u^2.$$

Definition 1. If a *bounded region* R of Σ is contained in a single coordinate neighbourhood $\phi(U)$ then we define the *area* of R to be

$$\iint_{\phi^{-1}(R)} |\boldsymbol{E_1} \times \boldsymbol{E_2}|\, du^1 du^2$$

We need to ensure that R is a set for which this definition of area makes sense: such sets are called bounded regions.

Definition 2. A *domain* of Σ is an open and connected subset whose boundary is the image of a circle under a regular smooth homeomorphism.

Definition 3. A *region* of Σ is the union of a domain with its boundary.

Definition 4. A subset of $\mathbb{R}^n$ is *bounded* if it is contained in some open ball in $\mathbb{R}^n$.

We need to show that our definition of area does not depend upon the choice of local parameterisation.

That is, if R is a bounded region in the regular surface Σ which is contained in the image of local parameterisations

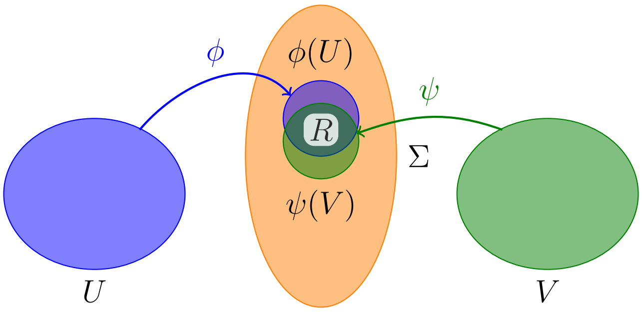
$$\begin{aligned} \phi : U &\to \Sigma \\ (u^1, u^2) &\mapsto \phi(u^1, u^2) \end{aligned}$$

and

$$\begin{aligned} \psi : V &\to \Sigma \\ (v^1, v^2) &\mapsto \psi(v^1, v^2) \end{aligned}$$

such that $\phi(u^1, u^2) = \psi(v^1, v^2)$, then we must prove that

$$\iint_{\phi^{-1}(R)} |\phi_{u^1} \times \phi_{u^2}| du^1 du^2 = \iint_{\psi^{-1}(R)} |\psi_{v^1} \times \psi_{v^2}| dv^1 dv^2.$$



The chain rule gives:

$$\begin{aligned}\psi_{v^1} &= \phi_{u^1}\frac{\partial u^1}{\partial v^1} + \phi_{u^2}\frac{\partial u^2}{\partial v^1}\\ \psi_{v^2} &= \phi_{u^1}\frac{\partial u^1}{\partial v^2} + \phi_{u^2}\frac{\partial u^2}{\partial v^2}.\end{aligned}$$

So writing

$$\frac{\partial(u^1,u^2)}{\partial(v^1,v^2)}$$

for the change of coordinates' Jacobian matrix,

$$\begin{aligned}\iint_{\psi^{-1}(R)} |\psi_{v^1}\times\psi_{v^2}|dv^1dv^2 &= \iint_{\psi^{-1}(R)} |\phi_{u^1}\times\phi_{u^2}|\left|\frac{\partial(u^1,u^2)}{\partial(v^1,v^2)}\right| dv^1dv^2\\ &= \iint_{\phi^{-1}(R)} |\phi_{u^1}\times\phi_{u^2}|du^1du^2.\end{aligned}$$

Active Learning

Question 5. Show that the area A of a bounded region R on the surface $z = f(x,y)$ is

$$A = \iint_{\phi^{-1}(R)} \sqrt{1+f_x^2+f_y^2}\,dx\,dy$$

where $\phi(x,y) = (x,y,f(x,y))$.

Answer 6.

$$\phi_x = (1,0,f_x),\ \phi_y = (0,1,f_y),$$

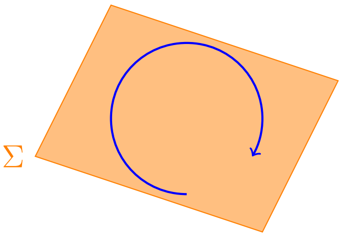
so

$$\phi_x\times\phi_y = \left|\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & f_x\\ 0 & 1 & f_y\end{pmatrix}\right| = (-f_x,-f_y,1).$$

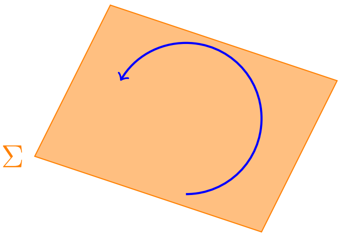
Hence

$$|\phi_x\times\phi_y| = \sqrt{f_x^2+f_y^2+1}.$$

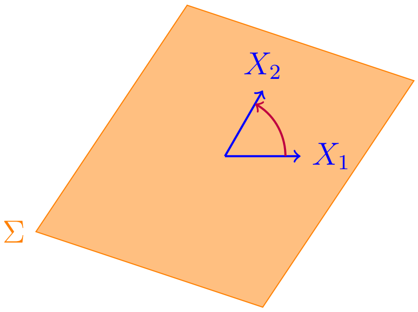
Intuitively, we orient a 2-dimensional vector space as either



or



More rigorously, an orientation on a plane is determined by an ordered basis (X_1, X_2).



Orientation given by shortest "direction" from X_1 to X_2.

Two ordered bases (X_1, X_2) and (Y_1, Y_2) determine the same orientation if and only if the change of basis matrix has positive determinant.

This suggests defining an equivalence relation:

Let (X_1, X_2) and (Y_1, Y_2) be ordered bases for the 2 dimensional vector space V.

Say that $(X_1, X_2) \sim (Y_1, Y_2)$ if the 2×2 matrix A defined by

$$(Y_1, Y_2) = (X_1, X_2)A$$

has positive determinant.

This is an equivalence relation:

1. reflexive: $(X_1, X_2) \sim (X_1, X_2)$;
2. symmetric: $(X_1, X_2) \sim (Y_1, Y_2)$ if and only if $(Y_1, Y_2) \sim (X_1, X_2)$;
3. transitive: if $(X_1, X_2) \sim (Y_1, Y_2)$ and $(Y_1, Y_2) \sim (Z_1, Z_2)$ then $(X_1, X_2) \sim (Z_1, Z_2)$.

Denote the equivalence class containing the ordered basis (X_1, X_2) by $[(X_1, X_2)]$.

Notice that are just two equivalence classes.

Active Learning

Question 7. Now let's make exactly the same definition for a n-dimensional vector space V, namely:

Let $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ be ordered bases for the n-dimensional vector space V, and say that $(X_1, \ldots, X_n) \sim (Y_1, \ldots, Y_n)$ if the $n \times n$ matrix A defined by

$$(Y_1, \ldots, Y_n) = (X_1, \ldots, X_n)A$$

has positive determinant.

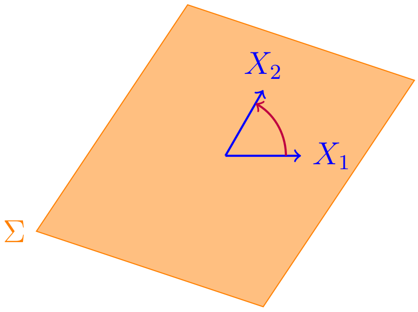
1. Is this an equivalence relation?
2. How many equivalence classes are there?

Answer 8. 1. Yes, exactly as above.

2. Two: the determinant is either positive or negative.

Definition 9. An *orientation* of a n-dimensional vector space V is a choice of one of the two possible equivalence classes of ordered basis for V under the equivalence relation $\sim$ defined above.

In particular, the bases



all belong to the same equivalence class,

If

$$\begin{aligned} \phi : U &\to \Sigma \\ (u^1, u^2) &\mapsto \phi(u^1, u^2) \end{aligned}$$

is a local parameterisation of the regular surface Σ near p, then the ordered basis $(\boldsymbol{E_1}, \boldsymbol{E_2}) = \left(\frac{\partial \phi}{\partial u^1}, \frac{\partial \phi}{\partial u^2}\right)$ determines an orientation on $T_p\Sigma$.

Another local parameterisation

$$\begin{aligned} \psi : V &\to \Sigma \\ (v^1, v^2) &\mapsto \psi(v^1, v^2) \end{aligned}$$

near p determines the same orientation if and only if the change of basis matrix at p,

$$\frac{\partial(v^1, v^2)}{\partial(u^1, u^2)} = \begin{pmatrix} \frac{\partial v^1}{\partial u^1} & \frac{\partial v^1}{\partial u^2} \\ \frac{\partial v^2}{\partial u^1} & \frac{\partial v^2}{\partial u^2} \end{pmatrix}$$

has positive determinant.

We would like to to extend the notion of orientation to a regular surface rather than just each tangent plane.

Definition 10. A regular surface Σ is *orientable* if it has an atlas (i.e. may be covered with coordinate neighbourhoods) so that whenever $p \in \Sigma$ lies in the image of two local parameterisations ϕ and ψ, the change of basis matrix at p has positive determinant. If this is possible, then the choice of such an atlas is called an *orientation* of Σ, and we say that Σ is *oriented*.

An orientation of a regular surface Σ can be thought of as an orientation of every tangent plane that "varies smoothly".

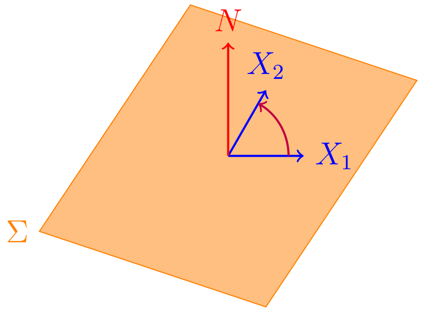
A surface which cannot be oriented is called *non-orientable*.

For a 2-plane in $\mathbb{R}^3$ (e.g. a tangent plane), an orientation is equivalent to a choice of a unit normal vector.

Given an orientation $[(X_1, X_2)]$, define

$$N := \frac{X_1 \times X_2}{|X_1 \times X_2|}$$

Two ordered bases determine the same orientation if and only if they define the same N.



Proposition 11. *A regular surface $\Sigma \subset \mathbb{R}^3$ is orientable if and only if there is a smooth map $N : \Sigma \to \mathbb{R}^3$ which assigns to each $p \in \Sigma$ a unit vector normal to Σ at p.*

Proof:

($\Rightarrow$) If Σ is orientable, then take an atlas whose change of coordinate matrices all have positive determinant. Then

$$\begin{aligned} N : \Sigma &\to \mathbb{R}^3 \\ p &\mapsto \frac{\boldsymbol{E_1} \times \boldsymbol{E_2}}{|\boldsymbol{E_1} \times \boldsymbol{E_2}|}(p) \end{aligned}$$

is well-defined (independent of local parameterisation) and smooth.

($\Leftarrow$) Suppose N is given, and take an atlas consisting of connected coordinate neighbourhoods.

Take $p \in \Sigma$, and $\phi : U \to \Sigma$ a local parameterisation near p. By exchanging u^1 and u^2 if necessary, we may ensure that

$$N(p) = \frac{\boldsymbol{E_1} \times \boldsymbol{E_2}}{|\boldsymbol{E_1} \times \boldsymbol{E_2}|}(p).$$

Since

$$\left\langle N, \frac{\boldsymbol{E_1} \times \boldsymbol{E_2}}{|\boldsymbol{E_1} \times \boldsymbol{E_2}|} \right\rangle$$

- is smooth (hence continuous), and
- is either 1 or -1,

it is constant, and since it is 1 at p, it is 1 throughout the connected set $\phi(U)$.

We can similarly ensure that

$$N = \frac{\boldsymbol{E_1} \times \boldsymbol{E_2}}{|\boldsymbol{E_1} \times \boldsymbol{E_2}|}$$

on every coordinate neighbourhood.

Then for any overlapping coordinate charts, $\phi_{u^1} \times \phi_{u^2}$ and $\phi_{v^1} \times \phi_{v^2}$ point in the same (rather than the opposite) direction, and so $\frac{\partial(u^1,u^2)}{\partial(v^1,v^2)}$ has positive determinant. $\square$