MATH 3968 Differential Geometry

Assignment, due in lecture, Tuesday 20th October

Second Semester, 2009

Whilst completing this assignment, you are welcome to consult the course notes and do Carmo, and to ask me questions. However your submitted assignment should be your own work; collaborating with classmates, or searching books/papers or online for assistance (except as noted below) are not permitted. This assignment is worth 20 % of your course grade.

1. **Relevant Background**

   The total curvature of a space curve

   $$\alpha : [a, b] \to \mathbb{R}^3$$

   is given by

   $$\int_a^b |k|.ds.$$

   Theorem 1 (Fenchel's Theorem, section 5.7, do Carmo)**.** *The total curvature of a simple closed curve is* $\geq 2\pi$*, with equality if and only if* α *is a convex planar curve.*

   I do not expect you to know a proof of this statement, but it is both beautiful and useful and you should remember it; you may need to use it.

   Some Motivation

   There are various analogues of total curvature for surfaces, a major theorem is the Gauss-Bonnet theorem which says in particular that the integral of the Gauss curvature over a compact closed surface is a topological invariant (you can think of a closed compact surface as one that has no boundary and is contained in a finite region of space; I mention them here for motivation but they are not needed to do this exercise). So to obtain something that gives non-topological information, it is reasonable to look at the integral of the mean curvature. In fact it is more natural to consider the integral of the square of the mean curvature, as this is invariant under so-called conformal changes of metric, namely those which preserve angles. This is called the *Willmore energy* of the (compact) surface,

$$W(\Sigma) = \int_\Sigma H^2 dA.$$

The *Willmore Conjecture* (posed in 1965) is that for all immersed tori (i.e. surfaces homeomorphic to the surface of a doughnut, possibly with self intersections),

$$W(\Sigma) \geq 2\pi^2$$

with equality if and only if Σ is obtained by taking a circle α of radius $\sqrt{2}r$ and moving a circle of radius r along it in such a way that the centre of the smaller circle moves along α, and the smaller circle is always in the normal plane of α. Attempts to solve this conjecture have led to important new developments in differential geometry; it is still unsolved although it is being actively pursued.

Question

Let $\alpha : [a, b] \to \mathbb{R}^3$ be a regular smooth parameterised closed curve. I will write α for both this curve and its image. Generate a surface (a potentially rather wobbly torus) by moving a circle along the curve α in such a way that the centre of the circle traces out α, and the circle is always in the normal plane of α (i.e. the plane generated by n and b). Assume that the resulting surface Σ is regular and in particular has no self intersections.

(a) Prove that

$$\int_\Sigma H^2 dA \geq 2\pi^2.$$

You may make use of a computer algebra package or online integration web site to evaluate any unpleasant integrals; or you may find it helpful to know that

$$\int_0^{2\pi} \frac{(1 - 2kr\cos\theta)^2}{4r(1 - kr\cos\theta)} d\theta = \frac{\pi}{2r\sqrt{1 - k^2r^2}}.$$

(b) Show that you obtain equality above if and only if α is a circle whose radius is $\sqrt{2}$ times the radius of the circle you move along α.

[12 marks]

2. Let $\alpha : (-c, c) \to \Sigma$, $\beta : (-c, c) \to \Sigma$ be regular curves on a regular oriented surface Σ with the property that

$$k_n^\alpha(t) = \pm k^\alpha(t),\ k_n^\beta(t) = \pm k^\beta(t) \text{ for all } t \in (-c, c),$$

where k_n denotes normal curvature, k denotes curvature and the superscript refers to the curve in question. Such curves are called geodesics. Assume that α and β intersect orthogonally at $p = \alpha(0) = \beta(0) \in \Sigma$. Show that

$$\tau^\alpha(0) = -\tau^\beta(0),$$

where τ denotes torsion.

[8 marks]