MATH3968 Assignment, Solutions

October 29, 2009

1. Parameterise the tube Σ using

$$x(s,\theta) = \alpha(s) + r\cos\theta\boldsymbol{n} + r\sin\theta\boldsymbol{b},$$

where $\boldsymbol{n}$, $\boldsymbol{b}$ are the unit normal and binormal of α (and $\boldsymbol{t}$ is the unit tangent vector), s is the arc-length of α, and r is the radius of the circle which is moved along α. Compute:

$$\begin{aligned} e_1 = \boldsymbol{x}_s &= (1 - kr\cos\theta)\boldsymbol{t} + r\tau\sin\theta\boldsymbol{n} - r\tau\cos\theta\boldsymbol{b} \\ e_2 = \boldsymbol{x}_\theta &= -r\sin\theta\boldsymbol{n} + r\cos\theta\boldsymbol{b}, \end{aligned}$$

and

$$N = \frac{e_1 \times e_2}{|e_1 \times e_2|} = -\cos\theta\boldsymbol{n} - \sin\theta\boldsymbol{t}$$

where k denotes the curvature and τ the torsion of α. Use

$$g_{ij} = < e_i, e_j >,\ h_{ij} = - < N_{u_i}, e_j >$$

to compute that with respect to the given coordinate basis,

$$[I] = \begin{bmatrix} (1 - kr\cos\theta)^2 + r^2\tau^2 & -r^2\tau \\ -r^2\tau & r^2 \end{bmatrix},$$

$$[II] = \begin{bmatrix} k\cos\theta(kr\cos\theta - 1) + r\tau^2 & -r\tau \\ -r\tau & r \end{bmatrix}.$$

Notice that in this example you want to compute the second fundamental form using $h_{ij} = - < N_{u_i}, \boldsymbol{x}_{u_j} >$, not $h_{ij} = < N, \boldsymbol{x}_{u_i u_j} >$, as otherwise the expression will involve derivatives of the curvature and torsion (it would still be correct, but much harder to compute).

Then use $[dN]^t = -[II][I]^{-1}$ and $H = \frac{1}{2}\text{trace}(dN)$ to obtain

$$H^2 = \left(\frac{1 - 2rk\cos\theta}{2r(1 - kr\cos\theta)}\right)^2,$$

and

$$\sqrt{\det g} = r(1 - kr\cos\theta).$$

Alternatively, you may use the formula

$$H = \frac{h_{11}g_{22} - 2h_{12}g_{12} + h_{22}g_{11}}{2(g_{11}g_{22} - g_{12}^2)},$$

where the g_{ij} are the entries in $[I]$ and the h_{ij} are the entries in $[II]$.

We assumed that Σ has no self intersections, which is equivalent to the radius of curvature of α always being greater than the radius r of the circle that is moved along it, i.e. $\frac{1}{|k(s)|} > r$ and so $1 - kr\cos\theta > 0$. Thus

$$\begin{aligned} \iint_\Sigma H^2 d\sigma &= \int_a^b \int_0^{2\pi} = \frac{(1-2rk\cos\theta)^2}{4r(1-kr\cos\theta)} d\theta ds \\ &= \int_a^b \frac{\pi}{2r\sqrt{1-(k(s)r)^2}} ds \\ &= \int_a^b \frac{|k(s)|\pi}{2|k(s)|r\sqrt{1-(k(s)r)^2}} ds. \end{aligned}$$

Now the function

$$\frac{1}{x\sqrt{1-x^2}}$$

takes its minimum value of 2 at $x = 1/\sqrt{2}$, so

$$\begin{aligned} \iint_\Sigma H^2 d\sigma &\geq \int_a^b \pi|k|ds \\ &\geq 2\pi^2 \end{aligned}$$

by Fenchel's Theorem.

[10 points]

2. We will have

$$\iint_\Sigma H^2 d\sigma = 2\pi^2$$

if and only if we have equality in both of the inequalities used above, i.e. if and only if $|k|r \equiv 1/\sqrt{2}$ and (by Fenchel's Theorem), α is a plane curve. Thus we have equality if and only if α is a circle of radius $\sqrt{2}r$, as required.

[2 points]

3. First note that as both curves are geodesics, $\boldsymbol{N} = \pm\boldsymbol{n}^\alpha$ along α, and similarly along β, and their orthogonal intersection gives $\boldsymbol{t}^\alpha = \pm\boldsymbol{b}^\beta$ and $\boldsymbol{t}^\beta = \pm\boldsymbol{b}^\alpha$ at 0. Thus

$$\begin{aligned} d\boldsymbol{N}_p(\boldsymbol{t}^\alpha(s)) &= d\boldsymbol{N}_p(\alpha'(s)) \\ &= (\boldsymbol{N} \circ \alpha)'(s) \\ &= \pm(\boldsymbol{n}^\alpha)'(s), \end{aligned}$$

with the sign of $(\boldsymbol{n}^\alpha)'(s)$ given by $\langle \boldsymbol{N}, \boldsymbol{n}^\alpha \rangle$. The same holds true for β. Thus we can write

$$d\boldsymbol{N}_p(\boldsymbol{t}^\alpha) = \langle \boldsymbol{N}, \boldsymbol{n}^\alpha \rangle (\boldsymbol{n}^\alpha)', \quad d\boldsymbol{N}_p(\boldsymbol{t}^\beta) = \langle \boldsymbol{N}, \boldsymbol{n}^\beta \rangle (\boldsymbol{n}^\beta)'.$$

The Frenet equation for the derivative of the normal

$$\boldsymbol{n}' = -k\boldsymbol{t} - \tau\boldsymbol{b}$$

and orthogonality of the tangents gives

$$\begin{aligned} \langle d\boldsymbol{N}_p(\boldsymbol{t}^\alpha), \boldsymbol{t}^\beta \rangle &= \langle \langle \boldsymbol{N}, \boldsymbol{n}^\alpha \rangle (\boldsymbol{n}^\alpha)', \boldsymbol{t}^\beta \rangle \\ &= -\tau^\alpha \langle \langle \boldsymbol{N}, \boldsymbol{n}^\alpha \rangle \boldsymbol{b}^\alpha, \boldsymbol{t}^\beta \rangle \\ &= -\tau^\alpha \langle \langle \boldsymbol{N}, \boldsymbol{n}^\alpha \rangle (\boldsymbol{t}^\alpha \times \boldsymbol{n}^\alpha), \boldsymbol{t}^\beta \rangle \\ &= -\tau^\alpha \langle (\boldsymbol{t}^\alpha \times \boldsymbol{N}), \boldsymbol{t}^\beta \rangle \end{aligned}$$

and similarly $\langle d\boldsymbol{N}_p(\boldsymbol{t}^\beta), \boldsymbol{t}^\alpha \rangle = -\tau^\beta \langle (\boldsymbol{t}^\beta \times \boldsymbol{N}), \boldsymbol{t}^\alpha \rangle$. As $d\boldsymbol{N}_p$ is self-adjoint, we have

$$\langle d\boldsymbol{N}_p(\boldsymbol{t}^\alpha), \boldsymbol{t}^\beta \rangle = \langle d\boldsymbol{N}_p(\boldsymbol{t}^\beta), \boldsymbol{t}^\alpha \rangle$$

and since

$$\langle (\boldsymbol{t}^\alpha \times \boldsymbol{N}), \boldsymbol{t}^\beta \rangle = -\langle (\boldsymbol{t}^\beta \times \boldsymbol{N}), \boldsymbol{t}^\alpha \rangle$$

the result follows. Note: The self-adjointness property may be utilised to give a different proof, whereby an explicit matrix is computed for $d\boldsymbol{N}_p$. In this case, self-adjointness ensures the matrix is symmetric about the main diagonal.

[8 points]