MATH3968 Lecture 1

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28 July 2009

- What is differential geometry and why is it useful?
- Course information
- Parameterised curves

What Is Differential Geometry?

Tough question. An attempt:

Differential geometry is

- the study of shape
- the study of smooth curved objects
- calculus on curved spaces

Curvature

Curvature seems important in these notions. What is it?

We would agree that a sheet of paper is flat.

The curvature of a piece of paper should be 0.

If we roll it up into say a cylinder or a cone, does its curvature change?

We can make the piece of paper "curvy" but there is always a direction in which the paper is straight.

So does the rolled up piece of paper have 0 curvature?

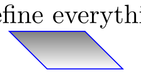
For surfaces we will define two notions of curvature.

One of these changes when we roll the piece of paper; the other one doesn't.

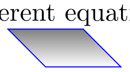
We prefer notions that don't change under distance-preserving transformations like "rolling up". They are *intrinsic*.

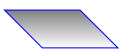
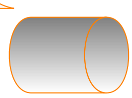
Extrinsic vs Intrinsic, Local vs Global

To begin with, we will define everything extrinsically.

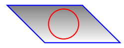
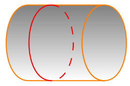
 extrinsically different from 

They are described by different equations.

 intrinsically the same as 

 extrinsically and intrinsically different from 

For example, in the cylinder we can draw a closed loop which we cannot contract to a point.

Differential geometry has two flavours: beautiful intrinsic constructions/results, and coordinate-based computations.

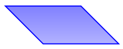
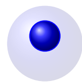
Geometry vs Topology

Have you ever tried to wrap a spherical present, such as a soccer ball?

Even locally, a sphere is intrinsically different to a plane (e.g. the sphere has positive curvature).

You cannot wrap even part of a sphere without folding the wrapping paper.

However,

 is topologically equivalent to 

Topology allows wrapping "paper" made from stockings.

What is Differential Geometry good for?

Where shall I start? (or stop!)

Optimisation Problems: it teaches us to do calculus on anything smooth. Examples:

- shortest path between two points: what path should a plane follow from Sydney to San Francisco?
- least energy: what shape soap film would you get if you dip a closed wire loop in soapy water? What path should the space shuttle take to Mars?

Applications: Physics

Differential geometry was/is used

- by Einstein to generalise special relativity to the case where the frames of reference are accelerating. He interpreted gravity as curvature.
- to estimate the mass of black holes
- in attempts to reconcile general relativity and quantum mechanics.

Applications: Mechanical Systems

- harmonic oscillator (e.g. pendulum)
- motion of particles within crystals
- What movements should a diver make to follow a particular path? Remember THAT Matthew Mitcham dive?

Connections with other mathematics?

If we agree the four basic pillars of mathematics are

Analysis, topology, geometry and algebra:

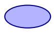
Analysis: huge overlap. Optimisation usually means solving differential equations, for which we need analysis.

Topology: topology underlies geometry.

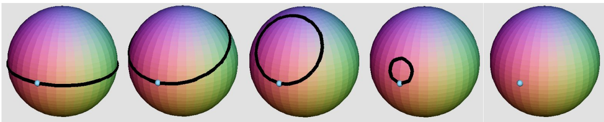
The Poincaré conjecture was perhaps THE outstanding problem in topology, until it was solved in 2002/2003 by Perelman using differential geometry.

Poincaré conjecture (theorem!)

Suppose M is a surface which

1. is connected
2. is bounded (contained in some finite box)
3. has no boundary e.g. , not 
4. is such that any closed loop can be continuously deformed to a point

then M is a (possibly stretched) sphere.



Poincaré says that the same is true for three-dimensional spaces.

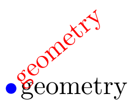
Algebra

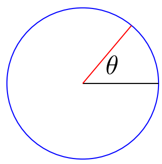
Much overlap, especially with Lie groups, which have both an algebraic and a geometric structure. Algebra: group; geometry: curved space.

For example, the space of linear transformations of $\mathbb{R}^2$ which preserve length and orientation is a Lie group, denoted by $SO(2)$.

It is the space of rotation matrices,

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\} \cong S^1 \quad \text{(a circle)}$$





Course Information

Required Texts
- "Differential Geometry of Curves and Surfaces", by Manfredo do Carmo, available at the student Co-op.
- "Lecture Notes for MATH 3968", by Nigel O'Brien, available at KopyStop.

Recommended Text "Differential Forms and Applications", by Manfredo do Carmo, available at the student Co-op

Office Hours Carslaw 723 Thursdays 1-2 p.m.

Assignments There will be two assignments, due at the beginning of class on Tuesday 8th September and on Tuesday 20th October respectively.

Expectations

Keep up to Date Look over your lecture notes as soon as possible after class, and definitely before the next lecture.

Do the Problems Mathematics is not a spectator sport. Doing problems is even more important than learning theory.

Ask Questions I don't acknowledge the concept of a "stupid question". Confused? Ask!

Tutorials

Each Wednesday afternoon tutorial will cover material from the Tuesday of that week, and the Wednesday and Thursday of the previous week.

You should have attempted all of the tutorial problems BEFORE coming to the tutorial.

Assessment

Final Exam 70%

Assignments 25% There will be two assignments, worth 12.5% each

Participation in Tutorials 5% To maximise your participation grade:

- make a solid attempt at the tutorial problems before each class.
- contribute to to the tutorial discussion.

The participation grade is based on effort, not on brilliance.

Objectives

This unit aims to give a hands-on introduction to differential geometry, in particular to

- introduce important notions such as curvature through the theory of curves and surfaces in 3-space
- motivate and introduce more abstract notions such as that of a manifold.

Outcomes

Students who successfully complete this unit should be able to:

1. describe and compute fundamental properties of a curve, such as curvature and torsion;
2. compute the mean and Gauss curvature of a surface in 3-space;
3. both describe geometrically what it means to differentiate "along a surface" and be able to compute this; in particular, to find geodesics on a given surface;
4. understand various ways in which calculus on flat spaces can be extended to calculus on curved spaces;
5. understand and use major theorems such as the Gauss-Bonnet theorem, that describe the global geometry of a surface in terms of its topology;
6. give examples of special types of surfaces, such as minimal surfaces, and analyse their properties;
7. work with abstract manifolds such as hyperbolic and projective spaces, and the generalisation of concepts such as covariant differentiation to the spaces;
8. define and compute integrals over manifolds.