MATH3968 Lecture 6

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Definition 1. A *parametrised surface* in $\mathbb{R}^3$ is a smooth mapping

$$\phi : U \to \mathbb{R}^3$$

where U is an open set in $\mathbb{R}^2$. ϕ is *regular* if

$$d\phi_p : \mathbb{R}^2 \to \mathbb{R}^3$$

is *injective* for all $p \in U$, ie $\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v}$ are linearly independent where in terms of coordinates, ϕ takes the form

$$\phi(u, v) = \left(\phi^1(u, v), \phi^2(u, v), \phi^3(u, v)\right) .$$

The regularity condition guarantees the existence of a tangent plane at each point.

Unlike with curves, this will not be the basic definition of "surface" that we shall use.

The reason is that it does not allow us to deal properly with global properties of surfaces.

Indeed, one of the most basic surfaces—the sphere—is not properly treated with this definition, because there is not a regular ϕ whose image is the whole sphere.

We need to be able to use more than one such map ϕ in order to cover the sphere.

Definition 2. Let $U \subset \mathbb{R}^n$ be open. We say that $\phi : U \to \mathbb{R}^m$ is a *homeomorphism* onto its image if it is continuous and has continuous inverse.

(Note that this requires ϕ to be one-to-one.)

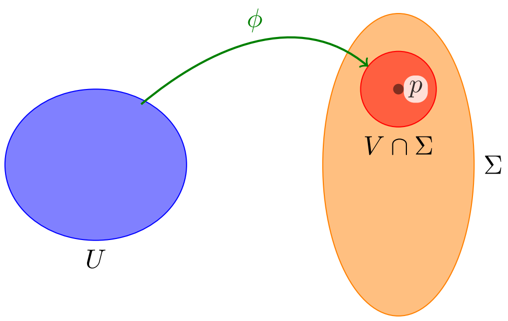
Definition 3 (Regular Surface)**.** A subset $\Sigma \subset \mathbb{R}^3$ is a *regular surface* if for each $p \in \Sigma$ there exists a neighbourhood V of p in $\mathbb{R}^3$ and a map

$$\phi : U \to V \cap \Sigma$$

of an open set $U \subset \mathbb{R}^2$ onto $V \cap \Sigma$ so that

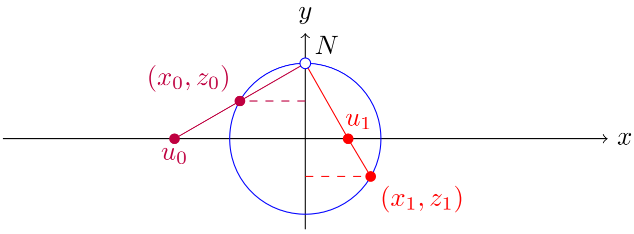
1. ϕ is smooth;
2. ϕ is a homeomorphism;
3. for each $q \in U$, $d\phi_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one (regularity condition).

ϕ is called a *local parameterisation* or *local coordinate* or *coordinate chart* near p, and $V \cap \Sigma$ is called a *coordinate neighbourhood* of p.



Example 4 (Stereographic Projection from the North Pole).

$$S^2 := \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$



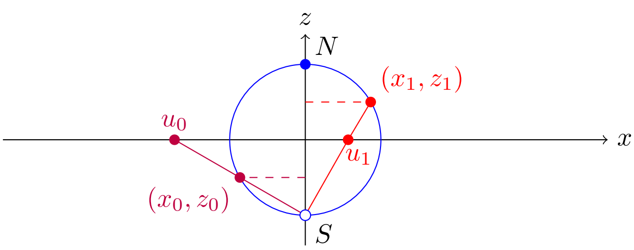
$$(u, v) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

Example 4 (continued). Use inverse of stereographic projection from the North Pole N to obtain a local parameterisation of $S^2 \setminus \{N\}$, given by

$$\begin{aligned} x &= \frac{2u}{u^2+v^2+1} \\ y &= \frac{2v}{u^2+v^2+1} \\ z &= \frac{u^2+v^2-1}{u^2+v^2+1} \end{aligned}$$

(exercise).

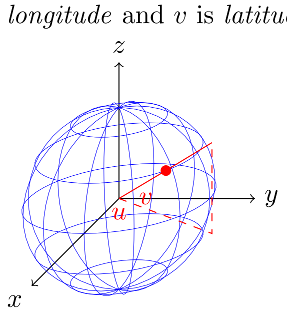
Example 4 (continued). Use inverse of stereographic projection from the South Pole S (or any point other than N) to obtain a local parameterisation near N.



Example 5. Alternatively, we can try

$$\phi(u, v) = (\cos u \cos v, \sin u \cos v, \sin v),$$

$0 < u < 2\pi$, $-\pi/2 < v < \pi/2$. Here u is *longitude* and v is *latitude*.



Note that $u = \theta$, $v = \pi/2 - \varphi$ where θ, φ are the usual spherical coordinates.

Example 5 (continued). ϕ is smooth and one-to-one, hence a homeomorphism if it is everywhere regular (Inverse Function Theorem).

It misses the North and South Poles, and the "international dateline".

The Jacobian matrix is

$$d\phi_{u,v} = \begin{pmatrix} -\sin u \cos v & -\cos u \sin v \\ \cos u \cos v & -\sin u \sin v \\ 0 & \cos v \end{pmatrix}.$$

The columns are linearly dependent only when $\cos v = 0$, which does not occur in our domain.

We can cover the whole sphere by, for example, interchanging the roles of x and z, and choosing a domain such that arcs "missed" by the two charts do not intersect.

Proposition 6. *Let $U \subset \mathbb{R}^2$ be open, and $f : U \to \mathbb{R}$ be smooth. Then the graph of f, namely*

$$\{(x, y, f(x, y)) : (x, y) \in U\}$$

is a regular surface.

Proof: Define

$$\begin{aligned} \phi &: U \to \mathbb{R}^3 \\ (u, v) &\mapsto (u, v, f(u, v)). \end{aligned}$$

1. ϕ is smooth.
2. ϕ is one-to-one, and its inverse is given by projection onto the first two co-ordinates, the restriction of a continuous map, and hence is continuous.
3. The first two rows of the Jacobian matrix $d\phi_{u,v}$ are the identity, so it is one-to-one.

□