MATH3968 Lecture 7

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Comment on Differentiating Norms

Recall that for $v(t) = (v^1(t), \ldots, v^n(t))$ and $w(t) = (w^1(t), \ldots, w^n(t))$,

$$v(t) \cdot w(t) = v^1(t)w^1(t) + \cdots v^n(t)w^n(t)$$

so by the product rule

$$\begin{aligned} \frac{d}{dt}(v(t) \cdot w(t)) &= (v^1)'(t) \cdot w^1(t) + v^1(t) \cdot (w^1)'(t) + \cdots \\ &\quad + (v^n)'(t) \cdot w^n(t) + v^n(t) \cdot (w^n)'(t) \\ &= v'(t) \cdot w(t) + v(t) \cdot w'(t). \end{aligned}$$

To differentiate $|v(t)|$, view it as $\sqrt{v(t) \cdot v(t)}$.

$$\begin{aligned} \frac{d}{dt}(|v(t)|) &= \frac{1}{2}(v(t) \cdot v(t))^{-\frac{1}{2}} \frac{d}{dt}(v(t) \cdot v(t)) \\ &= \frac{v'(t) \cdot v(t) + v(t) \cdot v'(t)}{2|v(t)|} \\ &= \frac{v'(t) \cdot v(t)}{|v(t)|} \end{aligned}$$

Active Learning

Question 1. Let $\alpha(t) : (a, b) \to \mathbb{R}^n$ be a parametrised curve which does not pass through the origin. Assume that $\alpha(t_0)$ is a point on the curve closest to the origin and that $\alpha'(t_0) \neq 0$. Show that $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Answer 2. Since the curve does not pass through the origin, $|\alpha(t)|^2$ is smooth on the open interval (a, b). Hence at the minimum of $|\alpha(t)|^2$ its derivative must vanish.

$$\begin{aligned} 0 &= \frac{d}{dt}|_{t=t_0}(\alpha(t) \cdot \alpha(t)) \\ &= 2\alpha'(t_0) \cdot \alpha(t_0) \end{aligned}$$

so $\alpha'(t_0)$ is orthogonal to $\alpha(t_0)$.

Recall from linear algebra that if

$$A : \mathbb{R}^m \to \mathbb{R}^n$$

is linear, then writing $[A]$ for the matrix of A with respect to the standard bases,

rank $[A] =$

$$\begin{aligned} &= \text{number of linearly independent columns of [A]} \\ &= \text{number of linearly independent rows of [A]} \\ &= \text{dimension of the image of } A \text{ (=column space of } [A]) \end{aligned}$$

To prove that these things are equivalent, reduce the matrix to row echelon form and check that

1. each of the above are invariant under row operations
2. the above are equal for a matrix in row echelon form

In particular, then

Proposition 3. *A linear map $A :: \mathbb{R}^{n+m} \to \mathbb{R}^n$ is surjective if and only if $[A]$ has rank n.*

Recall also the

Theorem 4 (Rank–Nullity Theorem)**.** *Suppose*

$$A : \mathbb{R}^m \to \mathbb{R}^n$$

is linear, then

$$\textit{dim image } (A) + \textit{dim ker } (A) = m$$

or

$$\textit{rank + nullity = number of columns.}$$

The rank–nullity theorem immediately gives us

Proposition 5. *A linear map $A : \mathbb{R}^n \to \mathbb{R}^{n+m}$ is injective if and only if $[A]$ has rank n, where $[A]$ is the matrix of A with respect to the standard bases.*

This proposition enables us in particular to check when the differential $d\phi_q$ is injective.

Definition 6 (Regular/Critical Values)**.** Let $F : U \subset \mathbb{R}^{n+m} \to \mathbb{R}^n$ be a smooth map on the open set U. $p \in U$ is a *critical point* of F if

$$dF_p : \mathbb{R}^{n+m} \to \mathbb{R}^n$$

fails to be surjective (onto). The image $F(p)$ of a critical point is called a *critical value* of F; a point in $\mathbb{R}^n$ which is not the image of any critical point is called a *regular value* of F.

Proposition 7. *If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.*

Proof: Use Implicit Function Theorem:

Take $p = (x_0, y_0, z_0) \in f^{-1}(a)$, and by renaming the axes if necessary assume that $\dfrac{\partial f}{\partial z}(p) \neq 0$.

By the Implicit Function Theorem, there are open neighbourhoods $U \subset \mathbb{R}^2$ of (x_0, y_0) and $V \subset \mathbb{R}$ of z_0 together with a smooth map

$$g : U \to V$$

such that $g(x_0, y_0) = z_0$ and

$$f(x, y, g(x, y)) = a$$

for all $(x, y) \in U$.

Near p, the surface $f^{-1}(a)$ is given by the graph of g, which we proved last lecture to be a regular surface.

That is, we checked that

$$\begin{array}{rccc} \phi : & U & \to & \mathbb{R}^3 \\ & (x, y) & \mapsto & (x, y, g(x, y)). \end{array}$$

gives a local coordinate about p.

Since $p \in f^{-1}(a)$ was arbitrary, $f^{-1}(a)$ is a regular surface. $\square$

Active Learning

Question 8. Let $f(x, y, z) = z^2$.

1. Is 0 a regular value of f?
2. Is $f^{-1}(0)$ a regular surface?

Answer 9.

1. $df_{(x,y,z)} = (0, 0, 2z)$, so $(0, 0, 0)$ is a critical point of f and hence $f(0, 0, 0) = 0$ is a critical value.
2. However
$$\{(x, y, z) \in \mathbb{R}^3 : z^2 = 0\}$$
describes the $x - y$ plane, which is a regular surface given by the parameterisation
$$\begin{array}{cccc} \phi : & \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (x, y) & \mapsto & (x, y, 0) \end{array}$$