MATH3968 Lecture 8

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Example 1 (Sphere). We can define the sphere implicitly as

$$\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=1\}.$$

It is the level set $F(x,y,z)=1$ of the function $F(x,y,z)=x^2+y^2+z^2$.

$$dF_{(x,y,z)}=(2x,2y,2z),$$

so the only critical point of F is $(0,0,0)$, and hence its only critical value is 0.

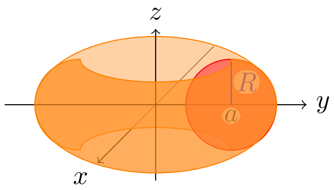
In particular, this shows that the sphere is a regular surface.

Example 2 (Torus). We can define a torus of internal radius R and external radius $a>R$ by rotating the circle

$$(y-a)^2+z^2=R^2$$

in the yz plane about the z axis, to give

$$\left(\sqrt{x^2+y^2}-a\right)^2+z^2=R^2.$$



Example 2 (continued). Let $f(x,y,z)=(\sqrt{x^2+y^2}-a)^2+z^2$; then

$$df_{(x,y,z)}=\left(\frac{2x(\sqrt{x^2+y^2}-a)}{\sqrt{x^2+y^2}},\frac{2y(\sqrt{x^2+y^2}-a)}{\sqrt{x^2+y^2}},2z\right).$$

The differential only fails to have maximal rank when $\sqrt{x^2+y^2}=a$ and $z=0$.

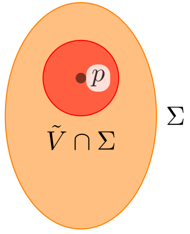
In particular, R^2 is a regular value of $f(x,y,z)=(\sqrt{x^2+y^2}-a)^2+z^2$, so the torus is a regular surface.

The Inverse Function Theorem also tells us that locally every regular surface can be realised as a graph:

Proposition 3. *Let $\Sigma\subset\mathbb{R}^3$ be a regular surface and $p\in\Sigma$. Then there is an open neighbourhood V of p in Σ so that V is the graph of a smooth function which has one of the following three forms:*

$$z=f(x,y),\ y=g(x,z),\ \textit{or}\ x=h(y,z).$$

Definition 4. By an *open neighbourhood V of p in Σ* we mean the intersection $\tilde{V}\cap\Sigma$ of Σ with a neighbourhood $\tilde{V}$ of p in $\mathbb{R}^3$.



Theorem 5 (Inverse Function Theorem)**.** *Let $W \subset \mathbb{R}^n$ be an open set, and*

$$\begin{aligned} W &\to \mathbb{R}^n \\ x = (x^1, \ldots, x^n) &\mapsto (f^1(x), \ldots, f^n(x)) \end{aligned}$$

be a smooth map. Suppose that at $a = (a_1, \ldots, a_n) \in W$,

$$df(a) := \begin{pmatrix} f_1^1(a) & f_2^1(a) & \cdots & f_n^1(a) \\ f_1^2(a) & f_2^2(a) & \cdots & f_n^2(a) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^n(a) & f_2^n(a) & \cdots & f_n^n(a) \end{pmatrix}$$

is invertible, where $f_j^i = \frac{\partial f^i}{\partial x_j}$.

Then there are open neighbourhoods U of a and V of $b = f(a)$ so that $f|_U : U \to V$ is invertible with smooth inverse f^{-1}.

Definition 6 (Diffeomorphism)**.** A map $f : U \to V$ between open sets is a *diffeomorphism* if it is smooth and has smooth inverse.

The conclusion of the inverse function theorem is that $f|_U$ is a diffeomorphism onto its image.

Proof of Proposition 62: Let

$$\begin{aligned} \phi : \quad U &\to W = \tilde{W} \cap \Sigma \\ (u, v) &\mapsto (x, y, z) = \phi(u, v) \end{aligned}$$

be a local coordinate near $p = \phi(u_0, v_0)$.

Then by the regularity condition, $d\phi_{(u_0,v_0)}$ has rank 2. Assume

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

is invertible (this will give us z as a function of (x, y)).

Let

$$\begin{aligned} \pi : \quad \mathbb{R}^3 &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto (x, y). \end{aligned}$$

$$d\pi_{(x,y,z)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ for all } (x, y, z),$$

so by the chain rule,

$$d(\pi \cdot \phi)_{(u_0,v_0)} = d\pi_{\phi(u_0,v_0)} \cdot d\phi_{(u_0,v_0)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

By the inverse function theorem, there is a neighbourhood $U_0 \subset U$ of (u_0, v_0) in $\mathbb{R}^2$ such that $\pi \cdot \phi$ is a diffeomorphism onto its image $V_0 = \pi \cdot \phi(U_0)$.

So for $(x, y) \in V_0$, we have u, v as smooth functions of x, y:

$$(u(x, y), v(x, y)) = (\pi \cdot \phi)^{-1}(x, y).$$

Then on $V = \phi(U_0)$,

$$(x, y, z) = (x, y, z(u(x, y), v(x, y))),$$

so V is the graph of $z \cdot (\pi \cdot \phi)^{-1} : (x, y) \mapsto z$.

The other cases are similar. $\square$

In the study of regular curves, there was a canonical parameterisation, namely parameterisation by arc-length.

There is no canonical way to give a a local parameterisation of a regular surface.

We will often have to make definitions using a local parameterisation, and it is important that the definition does not depend upon the choice of parameterisation.

For example,

Definition 7 (Smooth function)**.** Let $f : V \subset \Sigma\mathbb{R}$ be a function defined on an open subset of a regular surface Σ. We say that f is *smooth at* $p \in V$ if there is a local parameterisation $\phi : U \to \Sigma$ with $p \in \phi(U) \subset V$ such that $f \cdot \phi : U \to \mathbb{R}$ is smooth at $\phi^{-1}(p)$. f is *smooth* if it is smooth at all points in its domain.