MATH3968 Lecture 9

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We need to know that if p it in the image of two such local parameterisations, ϕ and ψ, then $f \circ \psi$ is smooth at $\psi^{-1}(p)$ if and only if $f \circ \phi$ is smooth at $\phi^{-1}(p)$. This follows from

Proposition 1 (Change of Parameters)**.** *Let Σ be a regular surface, and suppose that $p \in \Sigma$ is in the image of two local parameterisations, $\phi : U \subset \mathbb{R}^2 \to \Sigma$, and $\psi : V \subset \mathbb{R}^2 \to \Sigma$. Write $W = \phi(U) \cap \psi(V)$. Then*

$$\psi^{-1} \circ \phi : \phi^{-1}(W) \to \psi(W)$$

is a diffeomorphism.

Proof:

$\psi^{-1} \circ \phi$ is the composition of homeomorphisms and hence a homeomorphism.

We would like to argue similarly that it is a diffeomorphism, but ψ^{-1} is not defined on an open set in $\mathbb{R}^n$, so we have no definition of smoothness for it.

Currently $\psi : V \underset{\text{open}}{\subset} \mathbb{R}^2 \to \Sigma \underset{\text{not open}}{\subset} \mathbb{R}^3$

We will extend ψ to a map $V \times \mathbb{R} \underset{\text{open}}{\subset} \mathbb{R}^3 \to \mathbb{R}^3$ whose image is open in $\mathbb{R}^3$.

This extension will locally have smooth inverse.

Choose $r \in \phi^{-1}(W)$, and set $q = \psi^{-1} \circ \phi(r)$.

We will show that $\psi^{-1} \circ \phi$ is smooth at r.

Write $\psi(u, v) = \big(\psi^1(u, v), \psi^2(u, v), \psi^3(u, v)\big)$.

Renaming axes if necessary, we may assume

$$\frac{\partial(\psi^1, \psi^2)}{\partial(u, v)}(q) = \det \begin{pmatrix} \frac{\partial \psi^1}{\partial u}(q) & \frac{\psi^1}{\partial v}(q) \\ \frac{\partial \psi^2}{\partial u}(q) & \frac{\psi^2}{\partial v}(q) \end{pmatrix} \neq 0.$$

Define

$$\begin{array}{rccl} \Psi : & V \times \mathbb{R} & \to & \mathbb{R}^3 \\ & (u, v, t) & \mapsto & (\psi^1(u, v), \psi^2(u, v), \psi^3(u, v) + t) \end{array}$$

The last column of $d\Psi_{(u,v,t)}$ is $(0, 0, 1)^T$, so

$$\det d\Psi_{(u,v,t)} = \frac{\partial(\psi^1, \psi^2)}{\partial(u, v)}(q) \neq 0.$$

Hence, by the inverse function theorem, there are (open) neighbourhoods A of q and B of $p = \Psi(q, 0) = \psi(q)$ in $\mathbb{R}^3$ such that

$$\Psi : A \to B$$

is a diffeomorphism.

Since ψ is continuous, $\phi^{-1}(B) \subset U \subset \mathbb{R}^2$ is open

$\psi^{-1} \circ \phi|_{\phi^{-1}(B)} = \Psi^{-1} \circ \phi|_{\phi^{-1}(B)}$ is a combination of smooth maps and hence smooth, as required. $\square$

Definition 2. Let Σ be a regular surface and $W \subset \Sigma$ an open subset. A function $f : W \to \mathbb{R}$ is *smooth* at $p \in W$ if for some coordinate chart $\phi : U \subset \mathbb{R}^2 \to \Sigma$ with $p \in \phi(U) \cap W$, the composition $f \circ \phi|_{U \cap \phi^{-1}(W)}$ is smooth.

This definition is independent of the choice of local coordinate chart ϕ.

Since by the above proposition $\psi^{-1} \circ \phi$ is a diffeomorphism, $f \circ \phi|_{U \cap V}$ is smooth if and only if $f \circ \psi|_{U \cap V}$ is smooth.

Similarly,

Definition 3. A map $f : \Sigma_1 \to \Sigma_2$ is *smooth* at $p \in \Sigma_1$ if for local coordinates $\phi_1 : U_1 \subset \mathbb{R}^2 \to \Sigma_1$ near p, and $\phi_2 : U_2 \subset \mathbb{R}^2 \to \Sigma_2$ near $f(p)$ with $f(\phi_1(U_1)) \subset U_2$,

$$\phi_2^{-1} \circ f \circ \phi_1$$

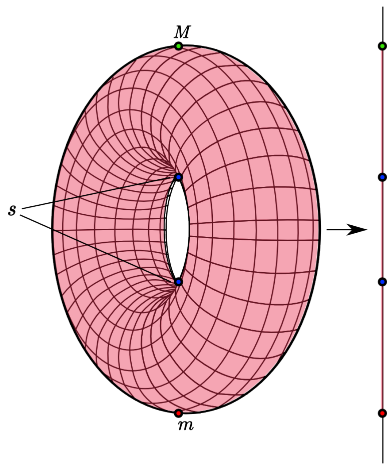
is smooth at $\phi_1^{-1} p$.

This definition is also independent of the choice of local coordinate systems.

$\psi_2^{-1} \circ f \circ \psi_1|_{U_1 \cap V_1}$ is smooth if and only if $\phi_2^{-1} \circ f \circ \phi_1|_{U_1 \cap V_1}$ is smooth, since both $\psi_1^{-1} \circ \phi_1$ and $\psi_2^{-1} \circ \phi_2$ are diffeomorphisms.

Thus we can extend our notion of diffeomorphism to maps between surfaces (they are smooth maps with smooth inverse) .

Example 4 (Height function on a torus).



How would you prove that these critical points are of the types claimed?

Example 5. Local coordinates are diffeomorphisms.

By definition, a local coordinate $\phi : U \to \Sigma$ is smooth and invertible.

For each $p \in U$, if $\psi : V \to \Sigma$ is another local coordinate about p then $\phi^{-1} \circ \psi|_{V \cap \psi^{-1}(U)}$ is smooth.

This is the definition of what it means for ϕ^{-1} to be smooth at p.

Active Learning

Question 6. Let Σ be the paraboloid $z = x^2 + y^2$.

1. Show that Σ is a regular surface.
2. Show that Σ is diffeomorphic to a plane (that is, there is a diffeomorphism between Σ and a plane).

Answer 7. 1. One answer: let $f(x, y, z) = z - x^2 - y^2$. Then $\Sigma = f^{-1}(0)$, and since

$$df_{(x,y,z)} = (-2x, -2y, 1)$$

has rank 1 for all $(x, y, z) \in \mathbb{R}^3$, Σ is the pre-image of a regular value of a smooth function, and hence a regular surface.

Alternatively (and this will answer both questions), consider

$$\begin{array}{rccl} \phi: & \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (u, v) & \mapsto & (u, v, u^2 + v^2). \end{array}$$

Answer 7 (continued)**.** We will show that ϕ is a homeomorphism., answering both parts of the question

1. ϕ is smooth.
2. ϕ has inverse $(x, y, z) \mapsto (x, y)$ which is the restriction of a continuous function and hence continuous. So ϕ is a homeomorphism.
3. $$d\phi_{(u,v)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2u & 2v \end{pmatrix}$$

   which has rank 2 for all $(u, v) \in \mathbb{R}^2$.

Thus $\phi : \mathbb{R}^2 \to \Sigma$ is a parameterisation of all of Σ, and hence a diffeomorphism.

A number of interesting surfaces can be obtained as surfaces of revolution.

Let Σ be the surface in $\mathbb{R}^3$ obtained by rotating the regular plane curve

$$x = f(v),\ z = g(v)$$

about the z-axis, where we assume that the curve does not intersect the z-axis.

$$\phi(u, v) = (f(v) \cos u, f(v) \sin u, g(v))$$

defines local coordinates. But by changing the range of angles (u, v) for which the map ϕ is defined we can cover the entire surface of revolution.

Example 8 (Torus). Let $x = a + b \cos v$, $z = b \sin v$, $b < a$.

$$\begin{array}{rccl} \phi: & (0, 2\pi) \times (0, 2\pi) & \to & \mathbb{R}^3 \\ & (u, v) & \mapsto & ((a + b \cos v) \cos u, (a + b \cos v) \sin u, b \sin v) \end{array}$$

is one local coordinate; together with

$$\begin{array}{rccl} \phi: & (-\frac{\pi}{2}, \frac{3\pi}{2}) \times (-\frac{\pi}{2}, \frac{3\pi}{2})) & \to & \mathbb{R}^3 \\ & (u, v) & \mapsto & ((a + b \cos v) \cos u, (a + b \cos v) \sin u, b \sin u) \end{array}$$

we have an atlas for the torus.