MATH3968 Lecture 10

Dr Emma Carberry

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Tangent Plane

Definition 1. Let $\Sigma \subset \mathbb{R}^3$ be a regular surface, and take $p \in \Sigma$. A *tangent vector* to Σ at p is the velocity vector $\alpha'(0)$ of some smooth parametrised curve $\alpha : (-\epsilon, \epsilon) \to \Sigma$ with $\alpha(0) = p$.

Definition 2. The set of all tangent vectors to Σ at p is called the *tangent plane to* Σ *at* p, and is denoted by $T_p\Sigma$.

Proposition 3. *Let* Σ *be a regular surface, and take* $p \in \Sigma$*. The tangent plane* $T_p\Sigma$ *is a 2-dimensional linear subspace of* $\mathbb{R}^3$*, and is equal to* $d\phi_q(\mathbb{R}^2)$*, for any local parameterisation* $\phi : U \subset \mathbb{R}^2 \to \Sigma$ *with* $\phi(q) = p$*.*

Proof: The first statement follows from the second one.

- $\mathbf{d}\phi_q(\mathbb{R}^2) \subset T_p\Sigma$:
  - Take $v \in \mathbb{R}^2$, and choose $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^2$ such that $\alpha(0) = q$ and $\alpha'(0) = v$.
  - By definition, $d\phi_q(v) = (\phi \circ \alpha)'(0)$, so $d\phi_q(v) \in T_p\Sigma$.
- $\mathbf{T}_p\Sigma \subset d\phi_q(\mathbb{R}^2)$
  - Take $X \in T_p\Sigma$, and choose $\alpha : (-\epsilon, \epsilon) \to \Sigma$ so that $\alpha(0) = p$, $\alpha'(0) = X$.
  - The local coordinate charts ϕ are diffeomorphisms, so $\phi^{-1} \circ \alpha : (-\epsilon, \epsilon) \to U$ are smooth curves.
  - $d\phi_q(\phi^{-1} \circ \alpha)'(0) = (\phi \circ \phi^{-1} \circ \alpha)'(0) = \alpha'(0)$, so $X \in \phi_q(\mathbb{R}^2)$. □

Definition 4. Given a local parameterisation $\phi : U \to \Sigma$ with $\phi(q) = p$, write e_1, e_2 for the standard basis of $\mathbb{R}^2$. Then

$$\phi_u(q) = \frac{\partial \phi}{\partial u}(q) = d\phi_q(e_1),\ \phi_v(q) = \frac{\partial \phi}{\partial v}(q) = d\phi_q(e_2)$$

is called the *basis of* $T_p\Sigma$ *associated to* ϕ, and if $X \in T_p\Sigma$ is given by

$$X = a\phi_u(q) + b\phi_v(q)$$

we call (a, b) the *coordinates* of X with respect to ϕ.

Take $X \in T_p\Sigma$, and let (a, b) be the coordinates of X with respect to ϕ, $\phi(q) = p$.

Let $\alpha : (-\epsilon, \epsilon) \to \Sigma$ be a smooth curve with $\alpha(0) = p$, $\alpha'(0) = X$.

Then writing $\phi^{-1} \circ \alpha(t) = (u(t), v(t))$,

$$\begin{aligned} a\phi_u(q) + b\phi_v(q) &= \alpha'(0) \\ &= (\phi \circ (\phi^{-1} \circ \alpha))'(0) \\ &= d\phi_q(u'(0)e_1 + v'(0)e_2) \\ &= u'(0)\phi_u(q) + v'(0)\phi_v(q) \end{aligned}$$

so

$$(a, b) = (u'(0), v'(0)).$$

Active Learning

Question 5. Let $\phi : U \subset \mathbb{R}^2 \to \Sigma \subset \mathbb{R}^3$ be a local parameterisation of a regular surface Σ, and for $q \in U$, consider

$$d\phi_q : \mathbb{R}^2 \to T_{\phi(q)}\Sigma.$$

What is the matrix of $d\phi_q$ with respect to

- the standard basis e_1, e_2 on $\mathbb{R}^2$;
- the basis $\phi_u(q), \phi_v(q)$ of $T_{\phi(q)}\Sigma$?

Answer 6. The identity matrix.

Differential of a smooth map

Definition 7. Let $f : \Sigma_1 \to \Sigma_2$ be a smooth map with $f(p_1) = p_2$. The *differential* df_{p_1} *of* f *at* p_1 is a linear map

$$df_{p_1} : T_{p_1}\Sigma_1 \to T_{p_2}\Sigma_2;$$

to define $df_{p_1}(X)$, take a smooth curve $\alpha_1 : (-\epsilon, \epsilon) \to \Sigma_1$ with $\alpha_1(0) = p_1$ and $\alpha_1'(0) = X$ and set

$$df_{p_1}(X) = (f \circ \alpha_1)'(0).$$

Proposition 8. *1. The differential df_{p_1} defined above is a well-defined linear map.*

2. For $i = 1, 2$, let ϕ_i, be a local parameterisation of Σ_i about p_i with $\psi_i(q_i) = p_i$. Writing $\phi_i^{-1} = (u_i, v_i)$, the matrix of df_{p_1} with respect to the basis $((\phi_i)_{u_i}, (\phi_i)_{v_i})$ is

$$\begin{pmatrix} \dfrac{\partial u_2}{\partial u_1}(q_1) & \dfrac{\partial u_2}{\partial v_1}(q_1) \\ \dfrac{\partial v_2}{\partial u_1}(q_1) & \dfrac{\partial v_2}{\partial v_1}(q_1) \end{pmatrix}$$

Proof: The first statement follows from the second.

$$\alpha_i : (-\epsilon, \epsilon) \to \Sigma_i$$

Take $X \in T_{p_1}\Sigma_1$, and let (a, b) be the coordinate of X with respect to ϕ_1.

Let $\alpha_1(-\epsilon, \epsilon) \to \Sigma_1$ be a smooth curve with $\alpha_1(0) = p_1$ and $\alpha_1'(0) = X$.

Then $\alpha_2 = f \circ \alpha_1$ is a smooth curve in Σ_2, and writing $\phi_i^{-1} \circ \alpha_i(t) = (u_i(t), v_i(t))$,

$$\begin{aligned} \alpha_2'(0) &= u_2'(0)(\phi_2)_{u_2} + v_2'(0)(\phi_2)_{v_2} \\ &= \left(\frac{\partial u_2}{\partial u_1}(q_1)u_1'(0) + \frac{\partial u_2}{\partial v_1}(q_1)v_1'(0)\right) \\ &\quad + \left(\frac{\partial v_2}{\partial u_1}(q_1)u_1'(0) + \frac{\partial v_2}{\partial v_1}(q_1)v_1'(0)\right). \end{aligned}$$

Observe the coordinates of $(f \circ \alpha_1)'(0)$ with respect to ϕ_2 depend only on the coordinates of $\alpha_1'(0)$ with respect to ϕ_1, and hence only on $X = \alpha_1'(0)$, not on the choice of α_1. □

Definition 9. A vector Y orthogonal to $T_p\Sigma$ is called a *normal vector* to the regular surface Σ and p.

In particular, given a local parameterisation $\phi : U \subset_{\text{open}} \to \Sigma$ near p with $\phi(q) = p$, we can choose between the two unit normal vectors to Σ and p by setting

$$N(q) = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|}(q)$$

Why do we define this map on U rather than on the surface Σ itself?

Definition 10. A *parametrised surface* is a smooth map $\phi : U \to \mathbb{R}^3$ where $U \subset \mathbb{R}^2$ is open.

If the differential $d\phi_{(u,v)}$ is not one-to-one (i.e., has rank < 2), we say that (u, v) is a *singular point* of ϕ.

If the differential $d\phi_{(u,v)}$ is one-to-one (i.e., has rank 2), we say that (u, v) is a *regular point* of ϕ.

The parametrised surface is *regular* if all $(u, v) \in U$ are regular points of ϕ.

Notice that we have NOT required that the map ϕ be one-to-one.

Example 11.

$$\begin{array}{cccc} \phi : & \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (u,v) & \mapsto & (u^2-1, v, u(u^2-1)) \end{array}$$

The differential at (u, v) is

$$d\phi_{(u,v)} = \begin{pmatrix} 2u & 0 \\ 0 & 1 \\ 3u^2-1 & 0 \end{pmatrix}$$

Since both $2u$ and $3u^2 - 1$ cannot be 0,

$$d\phi_{(u,v)} : \mathbb{R}^2 \to \mathbb{R}^3$$

is for each $(u, v) \in \mathbb{R}^2$ a one-to-one linear mapping.

Example 11 (continued)*.* However, the trace of this regular parametrised surface is NOT a regular surface.

ϕ is not one-to-one, since

$$(u_1^2 - 1, v_1, u_1(u_1^2 - 1)) = (u_2^2 - 1, v_2, u_2(u_2^2 - 1))$$

has solution $(u_1, v_1) = (1, v)$, $(u_2, v_2) = (-1, v)$, $v \in \mathbb{R}$.

For each $v \in \mathbb{R}$, there is no open neighbourhood V of $\phi(1, v) = (0, v, 0)$ in $\mathbb{R}^3$ such that $V \cap \phi(\mathbb{R}^2)$ can be parametrised by a coordinate chart—otherwise $T_{(0,v,0)}(\phi(\mathbb{R}^2))$ would exist and be a 2-dimensional linear subspace of $\mathbb{R}^3$, whereas we see that the space of velocity vectors to curves through $(0, v, 0)$ is given by