MATH3968 Lecture 11

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Example 1 (Torus). Recall that we made the torus T^2:

$$z^2 + \left(\sqrt{x^2+y^2} - a\right)^2 = b,\ b < a$$

into a regular surface by defining

$$\phi : (0, 2\pi) \times (0, 2\pi) \to R^3$$
$$(u, v) \mapsto ((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v)$$

as one local coordinate; together with

$$\psi : \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \to R^3$$
$$(u, v) \mapsto ((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v)$$

we have an atlas for the torus.

Example 1 (continued). Define $f : T^2 \to T^2$ to be the reflection in the yz-plane, namely $f(x, y, z) = (-x, y, z)$.

1. Describe $df_{(x,y,z)}$.
2. Calculate the matrix of the differential $df_{(\frac{a+b}{\sqrt{2}}, \frac{a+b}{\sqrt{2}}, 0)}$ with respect to the parameterisation ψ near $(\frac{a+b}{\sqrt{2}}, \frac{a+b}{\sqrt{2}}, 0)$ and $(-\frac{a+b}{\sqrt{2}}, -\frac{a+b}{\sqrt{2}}, 0)$.

Example 1 (continued). 1. Take $X \in T_{(x,y,z)}$ and let $\alpha : (-\epsilon, \epsilon) \to T^2$ be a smooth curve with $g(0) = (x, y, z)$, $\alpha'(0) = X$.

Let $R_x : \mathbb{R}^3 \to \mathbb{R}^3$ denote reflection in the yz-plane.

$$\begin{aligned} df_{(x,y,z)}(X) &= (f \circ \alpha)'(0) \\ &= \frac{d}{dt}\left(R_x \circ \alpha(t)\right)\big|_{t=0} \\ &= R_x(\alpha'(0)) \\ &= R_x(X). \end{aligned}$$

Example 1 (continued). 2. $(\frac{a+b}{\sqrt{2}}, \frac{a+b}{\sqrt{2}}, 0) = \psi(\frac{\pi}{4}, 0)$,$(-\frac{a+b}{\sqrt{2}}, -\frac{a+b}{\sqrt{2}}, 0) = \psi(\frac{5\pi}{4}, 0)$

Near $(\frac{a+b}{\sqrt{2}}, \frac{a+b}{\sqrt{2}}, 0)$,

$$\psi^{-1} \circ R_x \circ \psi(u, v) = (\pi - u, v)$$

which has differential

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Relevant Linear Algebra

Let V be a vector space over the real numbers.

Definition 2. A *bilinear form* B on V is a map $B : V \times V \to \mathbb{R}$ which is linear in each component, i.e.

1. $B(a_1 v_1 + a_2 v_2, w) = a_1 B(v_1, w) + a_2 B(v_2, w)$ for $a_1, a_2 \in \mathbb{R}$, $v_1, v_2, w \in V$, and
2. $B(v, a_1 w_1 + a_2 w_2) = a_1 B(v, w_1) + a_2 B(v, g_2 w_2)$ for $a_1, a_2 \in \mathbb{R}$, $v, w_1, w_2 \in v$.

Definition 3. The bilinear form B is *symmetric* if $B(v, w) = B(w, v)$ for all $v, w \in V$.

Definition 4. A bilinear form B on V is *positive definite* if $B(v, v) \geq 0$ for all $v \in V$, with equality if and only if $v = 0$.

Definition 5. A *inner product* on V is a positive definite symmetric bilinear form.

A *quadratic form* on $\mathbb{R}[v^1, \ldots, v^n]$ is a homogeneous polynomial of degree 2 in the variables $v^1, \ldots, v^n$.

We shall view them as maps:

Definition 6. A *quadratic form* on V is a map $Q : V \to \mathbb{R}$ such that

1. $Q(av) = a^2 Q(v)$ for all $a \in \mathbb{R}$ and $v \in V$, and
2. the map $B : V \times V \to \mathbb{R}$ defined by $B(v, w) = \frac{1}{2}(Q(v+w) - Q(v) - Q(w))$ is a (symmetric) bilinear form.

Conversely, a symmetric bilinear form B on V defines a quadratic form Q via

$$Q(v) = B(v, v).$$

Assume V is n-dimensional with basis $e_1, \ldots, e_n$, and write $v \in V$ as $v = \sum_{i=1}^n v^i e_i$.

A bilinear form B is represented with respect to this basis by a matrix A, where

$$B(v, w) = [v^1, \ldots, v^n] \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} w^1 \\ \vdots \\ w^n \end{bmatrix}.$$

B is symmetric if and only if the matrix A is symmetric: $A^t = A$.

The associated quadratic form Q is represented by the same matrix

$$Q(v) = [v^1, \ldots, v^n] \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$$

Note that $Q(v)$ is given by a homogeneous polynomial of degree 2 in the coefficients v^i.

Riemannian Metric

Let Σ be a regular surface, and

$$\begin{array}{cccc} \phi : & U & \to & \mathbb{R}^3 \\ & (u^1, u^2) & \mapsto & \phi(u^1, u^2) \end{array}$$

a local parameterisation near $p \in \Sigma$.

Notation

Write

$$\boldsymbol{E_1}(p) = \frac{\partial \phi}{\partial u^1}(\phi^{-1}(p)), \ \ \boldsymbol{E_2}(p) = \frac{\partial \phi}{\partial u^2}(\phi^{-1}(p)).$$

The restriction $\langle\cdot,\cdot\rangle_p$ of the standard inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^3$ to $T_p\Sigma \subset \mathbb{R}^3$ varies smoothly with p in the sense that the

$$g_{ij}(p) = \langle E_i(p), E_j(p)\rangle = E_i(p)\cdot E_j(p)$$

are smooth functions $\Sigma \to \mathbb{R}$.

We denote this inner product also by $g(p)(X,Y)$ or $X\cdot Y$, $X,Y \in T_p\Sigma$ and frequently omit the p.

Definition 7. We call the smoothly varying inner product $\langle\cdot,\cdot\rangle_p$ a *Riemannian metric* on Σ.

We shall often simply write $\langle\cdot,\cdot\rangle$.

Warning: A better name would be "Riemannian inner product". The word metric is traditional, but don't think of metric topology!

The 2×2 matrix

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

is symmetric and for $X = X^1\boldsymbol{E_1} + X^2\boldsymbol{E_2}, Y = Y^1\boldsymbol{E_1} + Y^2\boldsymbol{E_2} \in T_p\Sigma$, defines a smoothly varying inner product on the tangent spaces of Σ by

$$g(X,Y) = \langle X,Y\rangle = (X^1,X^2)\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \sum_{ij} g_{ij}v_iw_j$$

Definition 8. The associate quadratic form on $T_p\Sigma$ is denoted I_p and is called the *first fundamental form of the regular surface* Σ *at* p. The smoothly varying quadratic form I is called the *first fundamental form* of Σ.

The functions $g_{11}, g_{12} = g_{21}, g_{22} : \Sigma \to \mathbb{R}$ are called the *coefficients* of the first fundamental form.

What is the point?

The Riemannian metric (smoothly varying inner product) $g = \langle\cdot,\cdot\rangle$ and the first fundamental form (smoothly varying quadratic form) are equivalent.

Geometrically, they give us a notion of length and angle in every tangent plane.

This enables asked to define area.

It also enables us to distinguish between different "geometries" (more later).

Example 9. Let Σ be the plane in $\mathbb{R}^3$ through the point p_0 and containing the orthonormal vectors v, w.

Find the coefficients of the first fundamental form with respect to the global parameterisation $\phi(u^1,u^2) = p_0 + u^1v + u^2w$. For any $p \in \Sigma$, recall the notation

$$\boldsymbol{E_1}(p) = \frac{\partial\phi}{\partial u^1}(\phi^{-1}(p)), \quad \boldsymbol{E_2}(p) = \frac{\partial\phi}{\partial u^2}(\phi^{-1}(p)).$$

$$\boldsymbol{E_1}(p) = v,\ \boldsymbol{E_2}(p) = w$$

so since these are orthonormal,

$$\begin{aligned} g_{11}(\phi(u^1,u^2)) &= \langle\boldsymbol{E_1},\boldsymbol{E_1}\rangle = 1, \\ g_{12}(\phi(u^1,u^2)) &= \langle\boldsymbol{E_1},\boldsymbol{E_2}\rangle = 0 = \langle\boldsymbol{E_2},\boldsymbol{E_1}\rangle = g_{21}(\phi(u^1,u^2)), \\ g_{22}(\phi(u^1,u^2)) &= \langle\boldsymbol{E_2},\boldsymbol{E_2}\rangle = 1. \end{aligned}$$

If the coordinate chart ϕ is understood, we may write $\boldsymbol{E_1}, \boldsymbol{E_2}, g_{ij}$ directly as functions of (u^1,u^2).

Active Learning

Question 10. Let $v, w \in \mathbb{R}^3$ be orthonormal, l be the line with direction vector $v \times w$ through the point p_0, and Σ be the cylinder of radius 1 about the line l.The local parameterisations

$$\begin{array}{rlcl} \phi : & (0, 2\pi) \times \mathbb{R} & \to & \Sigma \\ & (u^1, u^2) & \mapsto & p_0 + \cos(u^1)v + \sin(u^1)w + u^2(v \times w) \end{array}$$

and

$$\begin{array}{rlcl} \psi : & (-\pi, \pi) \times \mathbb{R} & \to & \Sigma \\ & (u^1, u^2) & \mapsto & p_0 + \cos(u^1)v + \sin(u^1)w + u^2(v \times w) \end{array}$$

give Σ the structure of a regular surface.

Compute the coefficients of the first fundamental form with respect to these parameterisations.

Answer 11. Using the notation

$$\boldsymbol{E_1}(p) = \frac{\partial \phi}{\partial u^1}(\phi^{-1}(p)),\ \boldsymbol{E_2}(p) = \frac{\partial \phi}{\partial u^2}(\phi^{-1}(p)),$$

$$\boldsymbol{E_1}(\phi(u^1, u^2)) = \cos(u^1)w - \sin(u^1)v,\ \boldsymbol{E_2}(\phi(u^1, u^2)) = v \times w$$

so

$$g_{11}(\phi(u^1, u^2)) = \cos^2(u^1) + \sin^2(u^1) = 1,$$
$$g_{12}(\phi(u^1, u^2)) = 0,\ g_{22}(\phi(u^1, u^2)) = 1.$$

ψ is the same function with a different domain, so will give the same g_{ij}.

The geometric “reason” why we can find local parameterisations of the plane and the cylinder with the same g_{ij} is the fact that locally we can transform one into the other without any “stretching”.