MATH3968 Lecture 16

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We would like to compute the matrix of dN_p with respect to the basis ϕ_{u^1}, ϕ_{u^2}:

$$\mathbf{N}_{u^1} = \langle \mathbf{N}_{u^1}, \phi_{u^1} \rangle \phi_{u^1} + \langle \mathbf{N}_{u^1}, \phi_{u^2} \rangle \phi_{u^2},$$

$$\mathbf{N}_{u^2} = \langle \mathbf{N}_{u^2}, \phi_{u^1} \rangle \phi_{u^1} + \langle \mathbf{N}_{u^2}, \phi_{u^2} \rangle \phi_{u^2}.$$

We write

$$h_{ij} = - \langle \mathbf{N}_{u^i}, \mathbf{E}_j \rangle = - \langle dN(\mathbf{E}_i), \mathbf{E}_j \rangle .$$

We will not be able to compute some general expression for these inner product coefficients, but we can show that some of them are the same.

Differentiating

$$\langle \mathbf{N}, \phi_{u^1} \rangle = 0$$

with respect to u^2 gives

$$\langle \mathbf{N}_{u^2}, \phi_{u^1} \rangle + \langle \mathbf{N}, \phi_{u^1 u^2} \rangle = 0,$$

whereas differentiating

$$\langle \mathbf{N}, \phi_{u^2} \rangle = 0$$

with respect to u^1 gives

$$\langle \mathbf{N}_{u^1}, \phi_{u^2} \rangle + \langle \mathbf{N}, \phi_{u^1 u^2} \rangle = 0.$$

Hence

$$\langle \mathbf{N}_{u^2}, \phi_{u^1} \rangle = - \langle \mathbf{N}, \phi_{u^1 u^2} \rangle = \langle \mathbf{N}_{u^1}, \phi_{u^2} \rangle .$$

Recall that

$$h_{ij} = - \langle dN(\mathbf{E}_i), \mathbf{E}_j \rangle = - \langle \mathbf{N}_{u^1}, \mathbf{E}_j \rangle ,$$

which showed that

$$h_{21} = h_{12}.$$

Hence the matrix

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

is symmetric.

Take $\alpha(t) = \phi(u^1(t), u^2(t))$ with $\alpha(0) = p$. Then

$$\begin{aligned} dN_p(\alpha'(0)) &= dN_p((u^1)'(0)\phi_{u^1} + (u^2)'(0)\phi_{u^2}) \\ &= (u^1)'(0) dN_p(\phi_{u^1}) + (u^2)'(0) dN_p(\phi_{u^2}), \end{aligned}$$

but

$$dN_p(\phi_{u^1}) = \mathbf{N}_{u^1},\ dN_p(\phi_{u^2}) = \mathbf{N}_{u^2},$$

so

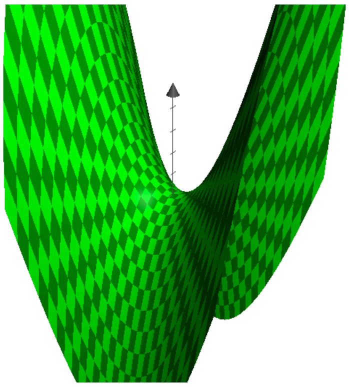
$$dN_p(\alpha'(0)) = (u^1)'(0)\mathbf{N}_{u^1} + (u^2)'(0)\mathbf{N}_{u^2}.$$

Example 1. Let Σ be the saddle-shaped surface

$$z = \frac{ax^2}{2} - \frac{ay^2}{2}$$

with parametrisation

$$\begin{aligned} \phi : \mathbb{R} \times \mathbb{R} &\to \Sigma \\ (u^1, u^2) &\mapsto \left(u^1, u^2, \frac{a(u^1)^2}{2} - \frac{a(u^2)^2}{2}\right) \end{aligned}$$



$$z = \frac{ax^2}{2} - \frac{ay^2}{2}$$

Example 1 (continued). A basis for the tangent space at $\phi(u^1, u^2)$ is

$$\boldsymbol{E_1} = (1, 0, au^1),\ \boldsymbol{E_2} = (0, 1, -au^2).$$

$$\boldsymbol{E_1} \times \boldsymbol{E_2} = \left|\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & au^1 \\ 0 & 1 & -au^2 \end{pmatrix}\right| = (-au^1, au^2, 1).$$

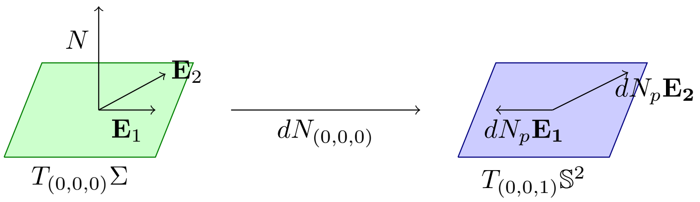
Hence the unit normal vector at $\phi(u^1, u^2)$ is

$$\mathbf{N}(\phi(u^1, u^2)) = \frac{1}{\sqrt{a^2u^1 + a^2u^2 + 1}}(-au^1, au^2, 1).$$

Example 1 (continued). Let's compute $-dN_p$ at $p = (0, 0, 0) = \phi(0, 0)$.

$$-dN_p(\phi_{u^1}) = -\mathbf{N}_{u^1}(0) = (a, 0, 0) = a\phi_{u^1}$$

$$-dN_p(\phi_{u^2}) = -\mathbf{N}_{u^2}(0) = (0, -a, 0) = -a\phi_{u^2}.$$



Example 1 (continued). So with respect to the basis $\phi_{u^1} = \boldsymbol{E_1}$, $\phi_{u^2} = \boldsymbol{E_2}$, the matrix of

$$-d\mathbf{N}_{(0,0,0)} : T_{(0,0,0)}\Sigma \to T_{(0,0,0)}\Sigma$$

is

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}.$$

So $-d\mathbf{N}_{(0,0,0)}$ has eigenspaces spanned by $\phi_{u^1} = \boldsymbol{E_1}$ and $\phi_{u^2} = \boldsymbol{E_2}$, with eigenvalues a and $-a$ respectively.

We defined

$$h_{ij} = -\langle dN(\mathbf{E_i}), \mathbf{E_j}\rangle = -\langle N_{u^i}, \mathbf{E_j}\rangle$$

and showed that the matrix

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

is symmetric.With respect to the orthonormal basis $\boldsymbol{E_1}, \boldsymbol{E_2}$, the bilinear form

$$B_p(X, Y) = -\langle dN_p(X), Y)\rangle$$

is also given by this 2×2 symmetric matrix.

Let $X = X_1\boldsymbol{E_1} + X_2\boldsymbol{E_2}, \;\; Y = Y_1\boldsymbol{E_1} + Y_2\boldsymbol{E_2} \in T_p\Sigma$.

$$\begin{aligned}
B_p(X, Y) &= -\langle dN_p(X), Y\rangle \\
&= -\langle dN_p(X_1\boldsymbol{E_1} + X_2\boldsymbol{E_2}), Y_1\boldsymbol{E_1} + Y_2\boldsymbol{E_2}\rangle \\
&= -\langle X_1 N_{u^1} + X_2 N_{u^2}, Y_1\boldsymbol{E_1} + Y_2\boldsymbol{E_2}\rangle \\
&= -(X_1, X_2)\begin{pmatrix} \langle N_{u^1}, \boldsymbol{E_1}\rangle & \langle N_{u^1}, \boldsymbol{E_2}\rangle \\ \langle N_{u^2}, \boldsymbol{E_1}\rangle & \langle N_{u^2}, \boldsymbol{E_2}\rangle \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\
&= (X_1, X_2)\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\
&= \sum_{ij} h_{ij} X_i Y_j
\end{aligned}$$

In particular the bilinear form B_p on $T_p\Sigma$ is symmetric

$$B_p(X, Y) = B_p(Y, X).$$

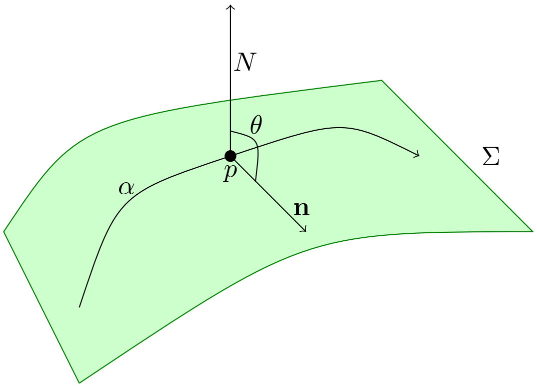
Definition 2. The corresponding quadratic form

$$II_p(X) = B_p(X, X) = -\langle dN_p(X), X\rangle$$

is called the *second fundamental form* of the oriented surface Σ.

There is a good geometric reason for the negative sign...

Let Σ be a regular surface with orientation N, and let $\alpha : (a, b) \to \Sigma$ be a regular curve, with $\alpha(0) = p \in \Sigma$. Recall the Frenet frame $\mathbf{t}, \mathbf{n}, \mathbf{b}$.



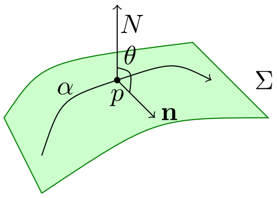
Definition 3. The *normal curvature* of α is

$$k_n(\alpha) = k\cos\theta,$$

where θ is the angle between N and n.

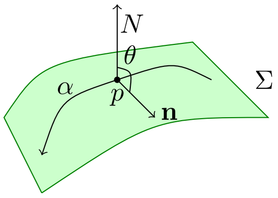
Geometrically, k_n is the projection of kn to N, with

- a positive sign if $\theta < \frac{\pi}{2}$
- a negative sign if $\theta > \frac{\pi}{2}$.



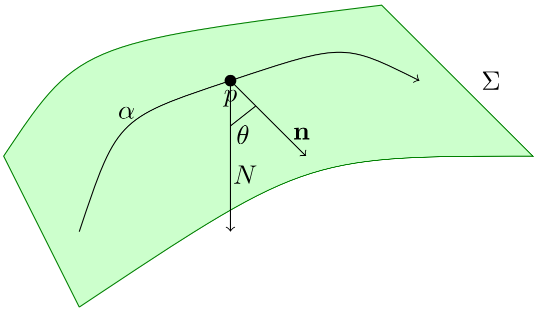
Changing the direction of α does not change n and hence does not change the normal curvature. This is true since parametrising $\alpha : (0, l) \to \Sigma$ with respect to arc-length s.

The same curve in the opposition direction is $\beta(l - s) = \alpha(s)$, so:



$$\begin{aligned}\frac{d^2\beta(l-s)}{d(l-s)^2} &= \frac{d(-\alpha(s))}{d(l-s)} \\ &= \frac{d^2\alpha(s)}{ds^2}\end{aligned}$$

However changing the orientation N changes the sign of the normal curvature:



Normal curvature is closely related to the second fundamental form:

Writing $N(s)$ for the restriction of N to $\alpha(s)$, and since

$$\langle N(s), \alpha'(s) \rangle = 0,$$

$$\langle N(s), \alpha''(s) \rangle = -\langle N'(s), \alpha'(s) \rangle$$

so

$$
\begin{aligned}
k_n(p) &= \langle N, kn \rangle (p) \\
&= \langle N(0), \alpha''(0) \rangle \\
&= -\langle N'(0), \alpha'(0) \rangle \\
&= -\langle dN_p(\alpha'(0)), \alpha'(0) \rangle \\
&= II_p(\alpha'(0))
\end{aligned}
$$

Notice that in particular, the normal curvature of α at $p = \alpha(0)$ depends only upon $\alpha'(0)$.