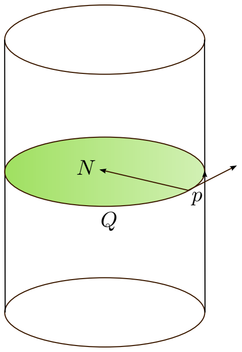
MATH3968 Lecture 17

Dr Emma Carberry

2 September 2009

Until further notice, let Σ be a regular orientable surface with orientation N. We observed

Proposition 1 (Meusnier)**.** *Let $p \in \Sigma$, and $X \in T_p\Sigma, |X| = 1$. Then all curves in Σ that have velocity vector X at p have normal curvature $k_n(p) = II_p(X)$ at p.*

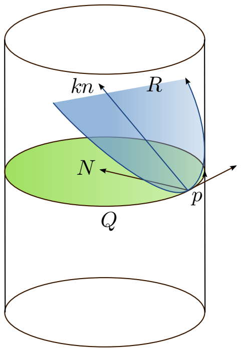


Proposition 2 (Meusnier)**.** *Take $p \in \Sigma$ and $X \in T_p\Sigma, |X| = 1$. Let Q be the plane spanned by X and N, and suppose the intersection of Σ with Q is the trace of a regular curve α. Let R be any of the planes through the point p which contain X and whose intersection with the surface is the trace of a regular curve β. Let $k^\alpha(p)$, $k^\beta(p)$ denote the curvatures of α and β, and $k_n^\alpha(p)$, $k_n^\beta(p)$ the normal curvatures. Then*

$$k^\alpha(p) = k^\beta(p) \cos\theta$$

where θ is the angle between the planes R and Q. Furthermore,

$$k_n^\beta(p) = II_p(X) = k_n^\alpha(p) = \pm k^\alpha(p).$$



Definition 3. We call the intersection of Σ with Q as above the *normal section* of Σ in the direction X; near p this defines a regular curve.

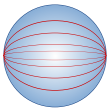
The normal curvature of such a curve at p is equal to its curvature at p up to sign (the sign depends upon the choice of N).

Example 4. Let Σ be a plane and let $p \in \Sigma$. All normal sections of Σ at p are straight lines, and hence have curvature 0. This corresponds to the fact that the second fundamental form is identically 0 on a plane.

Active Learning

Let Σ be a sphere of radius r, and let $p \in \Sigma$.

1. What are the normal sections of the sphere through p?



2. What are the curvatures of these curves? They are all $\frac{1}{r}$.

This corresponds to the fact that the second fundamental form of the sphere is identically $\frac{1}{r}$ on $\{X \in T_p\Sigma : |X| = 1\}$ for an inward pointing normal or identically $-\frac{1}{r}$ for an outward pointing normal.

Active Learning

Question 5. Let Σ be a cylinder of radius r, oriented with inward pointing normal N. Let $p \in \Sigma$. What are the maximum and minimum normal curvatures (values of the second fundamental form) at p, and in what directions do they occur? What is the angle between these directions?

Answer 6. The maximum normal curvature or $II_p(X)$, $|X| = 1$, is when X is horizontal. In this case, $II_p(X) = \frac{1}{r}$.

The minimum normal curvature or $II_p(X)$, $|X| = 1$, is when X is vertical. In this case, $II_p(X) = 0$.

The angle between the minimum and maximum directions is $\frac{\pi}{2}$.

This raises some questions:

- When the second fundamental form (normal curvature) does take different values in different directions, are the minimum and maximum directions unique (up to sign)? Yes.
- Is the angle between the minimum and maximum directions always $\frac{\pi}{2}$? Yes.
- Is there an easy way of writing down the value of the second fundamental form (normal curvature) in other directions in terms of these minimum and maximum values? Yes.

Theorem 7 (Euler). *If the normal curvatures $II_p(X)$, $X \in T_p\Sigma$, $|X| = 1$ are not all equal, then there is an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2$ of $T_p\Sigma$ such that*

1. $II_p(\mathbf{e}_1) = k_1 =$ *maximum value;*
2. $II_p(\mathbf{e}_2) = k_2 =$ *minimum value;*
3. $II_p(\cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2) = k_1 \cos^2\theta + k_2 \sin^2\theta$*.*

This theorem follows from linear algebra, specifically from the spectral theorem.

Theorem 8 (Spectral Theorem)**.** *A real symmetric $n \times n$ matrix can be diagonalised by an orthonormal basis of eigenvectors.*

Definition 9. Let V be a n-dimensional real vector space with inner product $\langle \cdot, \cdot \rangle$, and $A : V \to V$ a linear operator. We say that A is *self-adjoint* if for all $v, w \in V$,

$$\langle Av, w \rangle = \langle v, Aw \rangle .$$

This is equivalent to saying that the matrix of A with respect to an orthonormal basis is symmetric (exercise).

Theorem 10 (Spectral Theorem)**.** *Let V be a n-dimensional real vector space with inner product $\langle \cdot, \cdot \rangle$, and $A : V \to V$ a self-adjoint linear operator. There is an orthonormal basis of V consisting of eigenvectors of A.*

The fact that eigenvectors e_1 and e_2 of a self-adjoint linear operator A with distinct eigenvalues k_1 and k_2 must be orthogonal is elementary:

$$k_1 \langle e_1, e_2 \rangle = \langle Ae_1, e_2 \rangle = \langle e_1, Ae_2 \rangle = k_2 \langle e_1, e_2 \rangle$$

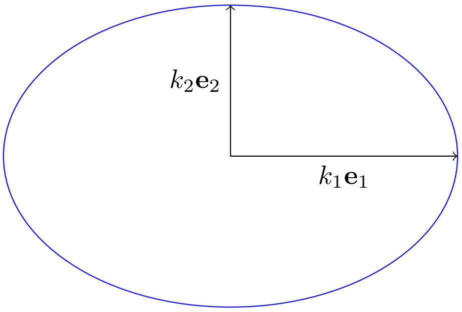
so since $k_1 \neq k_2$, we must have $\langle e_1, e_2 \rangle = 0$.

For a regular oriented surface Σ and $p \in \Sigma$, $dN_p : T_p\Sigma \to T_p\Sigma$ is a self-adjoint linear operator. Take an orthonormal basis of eigenvectors $\mathbf{e}_1, \mathbf{e}_2$ and without loss of generality assume the eigenvalues k_1, k_2 satisfy $k_1 \geq k_2$.

Then we may write any unit vector in $T_p\Sigma$ as $\cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2$ for some θ, and

$$\begin{aligned} II_p(\cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2) &= \langle dN_p(\cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2), (\cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2 \rangle \\ &= \langle k_1 \cos\theta \mathbf{e}_1 + k_2 \sin\theta \mathbf{e}_2, \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2 \rangle \\ &= k_1 \cos^2\theta + k_2 \sin^2\theta, \end{aligned}$$

which has maximum k_1 on $\pm\mathbf{e}_1$ and minimum k_2 on $\pm\mathbf{e}_2$.



Definition 11. We call $k_1(p)$, $k_2(p)$ the *principal curvatures* and $\mathbf{e}_1, \mathbf{e}_2$ the *principal directions* of Σ at p.