MATH3968 Lecture 18

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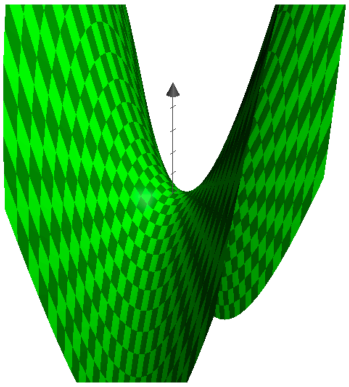
Example 1. We computed earlier that for the surface Σ given by

$$z = \frac{ax^2}{2} - \frac{ay^2}{2},$$



Example 1 (continued). ...if we orient Σ so that N points upwards at the saddle point, then the eigenvectors of $-dN_p$ at $p = (0,0,0)$ (the saddle point) are

$$\mathbf{e}_1 = (1,0,0),\ \mathbf{e}_2 = (0,1,0).$$

with eigenvalues a and $-a$ respectively. Note that these are orthogonal.

Definition 2. A point $p \in \Sigma$ at which $k_1(p) = k_2(p)$ is called an *umbilical point*.

We know two examples of surfaces at which every point is umbilical:

1. a plane;
2. a sphere.

Notice that throughout both of these surfaces, the principal curvature is constant.

Proposition 3. *If Σ is connected and every point in it is umbilical, then Σ is contained either in a sphere or a plane.*

Proof: Choose $p \in \Sigma$, and let $\phi : U \to \Sigma$ be a local parameterisation near p so that $\phi(U)$ is connected. The outline of our proof is:

1. $\phi(U)$ **is contained either in a sphere or a plane**, for which the steps are:
   (a) Show the principal curvatures $k := k_1 = k_2$ are constant throughout $\phi(U)$;
   (b) Using k and N, write down the equation of the plane ($k = 0$) or sphere ($k \neq 0$).
2. Choose another $q \in \Sigma$, and show that **it lies in the same sphere/plane as** p

To show: **the principal curvatures $k := k_1 = k_2$ are constant throughout $\phi(U)$**

Take $X = X_1\phi_{u^1} + X_2\phi_{u^2} \in T_p\Sigma$. Then

$$dN_p(X) = k(p)X,$$

that is,

$$X_1 N_{u^1} + X_2 N_{u^2} = kX_1\phi_{u^1} + kX_2\phi_{u^2}.$$

so

$$N_{u^1} = k\phi_{u^1},\, N_{u^2} = k\phi_{u^2}.$$

Differentiating the first equation with respect to u^2 and the second with respect to u^1,

$$N_{u^1u^2} = k_{u^2}\phi_{u^1} + k\phi_{u^1u^2},\, N_{u^1u^2} = k_{u^1}\phi_{u^2} + kX\phi_{u^1u^2}.$$

So comparing gives

$$k_{u^2}\phi_{u^1} = k_{u^1}\phi_{u^2},$$

so since ϕ_{u^1} and ϕ_{u^2} are linearly independent,

$$k_{u^1} = k_{u^2} = 0$$

and since $p \in \phi(U)$ was arbitrary, k is constant throughout $\phi(U)$.

To do: **using k and N, write down the equation of the plane ($k = 0$) or sphere ($k \neq 0$)**

If $k = 0$ throughout $\phi(U)$, then $N_{u^1} = dN(\boldsymbol{E_1}) = k\boldsymbol{E_1} = 0$, and similarly $N_{u^2} = 0$ in $\phi(U)$. Thus N is constant; $N(q) = N(p)$ for all $q \in \phi(U)$.

Then

$$\left\langle \phi(u^1, u^2), N(p) \right\rangle_{u^1} = \langle \phi_{u^1}, N \rangle = 0$$

and similarly

$$\left\langle \phi(u^1, u^2), N(p) \right\rangle_{u^2} = 0$$

so

$$\left\langle \phi(u^1, u^2), N(p) \right\rangle$$

is constant, which is the equation of a plane.

To do: **using k nd N, write down the equation of the plane ($k = 0$) or sphere ($k \neq 0$)**

If $k \neq 0$, then the "moving centre"

$$c(u^1, u^2) = \phi(u^1, u^2) - \frac{1}{k}N(u^1, u^2)$$

is fixed, which again we show by computing that derivatives with respect to u^1, u^2 are 0.

Then

$$\left\langle c, \phi(u^1, u^2) \right\rangle = \frac{1}{k^2}$$

so this is a sphere centred at c with radius $\frac{1}{k}$.

Choose another $q \in \Sigma$. We need to show that **it lies in the same sphere/plane as p.**

Since Σ is connected, we can take a path $\alpha : [0, 1] \to \Sigma$ from $p = \alpha(0)$ to $q = \alpha(1)$.

Each point $\alpha(t)$ in the trace of this curve is contained in some coordinate neighbourhood $\phi_a(U_a)$. The closed and bounded interval $[0, 1]$ is covered by the open sets $\alpha^{-1}\phi_a(U_a)$. By the Heine-Borel theorem, we can remove all but finitely many of these open sets and still cover $[0, 1]$.

We have then

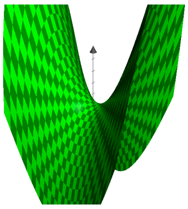
$$[0, 1] \subset \alpha^{-1}(\phi_1(U_1) \cup \ldots \cup \alpha^{-1}(\phi_n(U_n)),$$

and every one of the neighbourhoods $\phi_i(U_i)$ is contained in a plane or sphere. Clearly any two overlapping such neighbourhoods must be contained in the same plane/sphere, and since there are only finitely many of them we conclude that they are all contained in the same plane/sphere. $\square$

Definition 4. An *asymptotic direction* of Σ at p is a direction of $T_p\Sigma$ for which the normal curvature is 0. A curve $\alpha : (a, b) \to \Sigma$ is *asymptotic* if all the tangent lines are asymptotic directions.

Example 5. We computed earlier that for the surface Σ given by

$$z = \frac{ax^2}{2} - \frac{ay^2}{2}, \quad a > 0,$$



Example 5 (continued). ... if we orient Σ so that N points upwards at the saddle point, then the eigenvectors of $-dN_p$ at $p = (0, 0, 0)$ (the saddle point) are

$$\mathbf{e}_1 = (1, 0, 0), \quad , \mathbf{e}_2 = (0, 1, 0),$$

with eigenvalues a and $-a$ respectively. Then

$$II_p(\cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2) = a\cos^2\theta - a\sin^2\theta,$$

so asymptotic directions in $T_{(0,0,0)}\Sigma$ are when $\cos\theta = \pm\sin\theta$, so

$$\theta = \pm\frac{\pi}{4}, \quad \pm\frac{3\pi}{4}.$$

Example 5 (continued). The *Dupin indicatrix* at p is the set

$$\{X \in T_p\Sigma : II_p(X) = \pm 1\}.$$

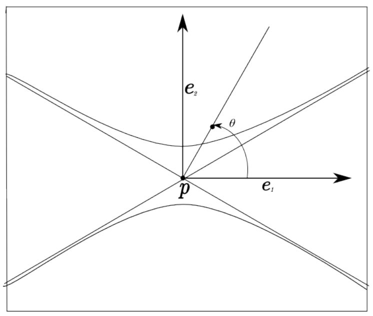
Writing $X = X_1\mathbf{e}_1 + X_2\mathbf{e}_2$, this is given by the equation

$$\pm 1 = II_p(X_1\mathbf{e}_1 + X_2\mathbf{e}_2) = X_1^2 II_p(\mathbf{e}_1) + X_2^2 II_p(\mathbf{e}_2) = X_1^2 k_1 + X_2^2 k_2$$

For this example, it is

$$X_1^2 - X_2^2 = \pm\frac{1}{a}.$$

Example 5 (continued). This is a pair of hyperbolae, whose asymptotes are the asymptotic directions.



We have shown that each tangent place $T_p\Sigma$ has an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2$ so that the matrix of $-dN_p : T_p\Sigma \to T_p\Sigma$ with respect to this basis is

$$\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

with $k_1 \geq k_2$.

There are two natural invariants of a 2×2 matrix, corresponding to the two symmetric functions of two variables:

Definition 6. The *Gauss curvature* of the oriented regular surface Σ at p is

$$K(p) = \det(-dN_p) = k_1(p)k_2(p),$$

and the *mean curvature* of Σ at p is

$$H(p) = \frac{1}{2}\mathrm{tr}(-dN_p) = \frac{k_1 + k_2}{2}.$$

Notice that for example when you "curl up" a piece of paper (i.e. part of the plane) into part of a cylinder, this changes the mean curvature but not the Gauss curvature.

As above, let Σ be a regular surface with orientation N.

We think of N both as

- an orientation on Σ (smooth field unit normal vectors)
- a smooth map $N : \Sigma \to \mathbb{S}^2$