MATH3968 Lecture 22

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Definition 1. A coordinate chart $\phi : U \subset \mathbb{R}^2 \to \Sigma \subset \mathbb{R}^3$ of the surface Σ is called *isothermal* or *conformal* if the vectors $\mathbf{E_1} = \phi_{u^1}$, $\mathbf{E_2} = \phi_{u^2}$ are orthogonal and have the same length, i.e.

$$g_{11} = g_{22}, \quad g_{12} = 0.$$

Theorem 2. *One can always cover a regular surface with isothermal coordinate charts.*

We will not prove this. □

Theorem 3. *If $\phi : U \to \mathbb{R}^3$ is a regular parameterised surface, and if ϕ is isothermal, then*

$$\Delta\phi := \phi_{u^1u^1} + \phi_{u^2u^2} = 2g_{11}HN$$

Thus such an ϕ is minimal if and only if

$$\Delta\phi := \phi_{u^1u^1} + \phi_{u^2u^2} = (0, 0, 0)$$

Functions satisfying $\Delta\phi = (0, 0, 0)$ are said to be *harmonic.*

Proof

Differentiating

$$\langle \phi_{u^1}, \phi_{u^1} \rangle = \langle \phi_{u^2}, \phi_{u^2} \rangle$$

with respect to u^1 gives

$$\langle \phi_{u^1u^1}, \phi_{u^1} \rangle = \langle \phi_{u^2u^1}, \phi_{u^2} \rangle .$$

Differentiating

$$\langle \phi_{u^1}, \phi_{u^2} \rangle = 0$$

with respect to u^2 gives

$$\langle \phi_{u^1u^2}, \phi_{u^2} \rangle + \langle \phi_{u^1}, \phi_{u^2u^2} \rangle = 0,$$

so combining with the above,

$$\langle \phi_{u^1u^1} + \phi_{u^2u^2}, \phi_{u^1} \rangle = 0.$$

Similarly,

$$\langle \phi_{u^1u^1} + \phi_{u^2u^2}, \phi_{u^2} \rangle = 0.$$

Hence $\phi_{u^1u^1} + \phi_{u^2u^2}$ is parallel to N. Since also

$$\langle N, \phi_{u^1u^1} + \phi_{u^2u^2} \rangle = h_{11} + h_{22},$$

we have that

$$\Delta\phi := \phi_{u^1u^1} + \phi_{u^2u^2} = (h_{11} + h_{22})N.$$

Since $g_{12} = 0, g_{22} = g_{11}$ the general expression

$$H = \frac{h_{11}g_{22} - 2h_{12}g_{12} + h_{22}g_{11}}{2(g_{11}g_{22} - g_{12}^2)}$$

for the mean curvature simplifies to

$$H = \frac{g_{11}(h_{11}+h_{22})}{2g_{11}^2} = \frac{h_{11}+h_{22}}{2g_{11}}.$$

So

$$\Delta\phi := \phi_{u^1u^1} + \phi_{u^2u^2} = 2g_{11}HN$$

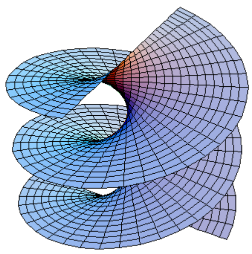
□

Active Learning

Question 4. Verify that the helicoid,

$$\phi(u^1,u^2) = (a\sinh u^2\cos u^1,\ a\sinh u^2\sin u^1,\ au^1)$$

is a minimal surface.



Answer 5.

$$\begin{aligned}\phi_{u^1} &= (-a\sinh u^2\sin u^1,\ a\sinh u^2\cos u^1,\ a),\\ \phi_{u^2} &= (a\cosh u^2\cos u^1,\ a\cosh u^2\sin u^1,\ 0)\end{aligned}$$

so

$$\begin{aligned}g_{11} &= a^2\sinh^2 u^2 + a^2 = a^2\cosh^2 u^2,\\ g_{12} &= 0,\\ g_{22} &= a^2\cosh^2 u^2 = g_{11}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\phi_{u^1u^1}+\phi_{u^2u^2} &= (-a\sinh u^2\cos u^1, -a\sinh u^2\sin u^1, 0)\\ &\quad +(a\sinh u^2\cos u^1, a\sinh u^2\sin u^1, 0)\\ &= = (0,0,0),\end{aligned}$$

and hence by the theorem above, the surface is minimal.

Comment for those who know some complex analysis:

- Minimal surfaces are strongly connected to complex analysis, essentially because the real and imaginary parts of analytic functions are harmonic.
- One can use this connection to write down (at least locally) many examples of minimal surfaces;

$$
\begin{aligned}
x(u^1,u^2) &= \mathrm{Re}\int f(1-g^2)dw \\
y(u^1,u^2) &= \mathrm{Re}\int f(1+g^2)dw \\
z(u^1,u^2) &= \mathrm{Re}\int 2fg\, dw
\end{aligned}
$$

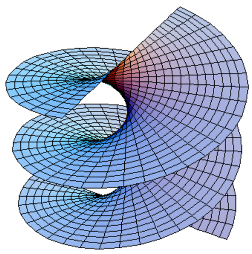
where $w = u^1 + iu^2$, and f, fg^2 are analytic whilst g is meromorphic. This is called the Weierstrass representation of the surface.

- Using $f(w) = 1$, $g(w) = w$ gives Enneper's surface, described below

Another example of a minimal surface is Enneper's surface

$$\phi(u^1,u^2) = \left(u^1 - \frac{(u^1)^3}{3} + u^1(u^2)^2, u^2 - \frac{(u^2)^3}{3} + u^2(u^1)^2, (u^1)^2 - (u^2)^2\right),$$

$(u^1,u^2) \in \mathbb{R}^2$.



Definition 6. A diffeomorphism

$$\psi : \Sigma_1 \to \Sigma_2$$

is an *isometry* if it preserves the inner product, i.e. if for all $p \in \Sigma_1$ and $X, Y \in T_p\Sigma_1$,

$$\langle X, Y\rangle_p = \langle d\psi_p(X), d\psi_p(Y)\rangle_{\psi(p)}\,.$$

We say then that the surfaces Σ_1 and Σ_2 are *isometric.*

Since the quadratic form $I(X) = \langle X, X\rangle$ uniquely determines and is uniquely determined by $\langle\cdot,\cdot\rangle$, it is equivalent to require that

$$I_p(X) = I_{\psi(p)}(d\psi_p(X)).$$

We also need the local version of this notion:

Definition 7. If for each $p \in \Sigma_1$ there is an open neighbourhood U_1 of p and a map

$$\psi : U_1 \to \Sigma_2$$

so that ψ is an isometry onto its image, then we say that Σ_1 *is locally isometric to* Σ_2. If it is also the case that Σ_2 is locally isometric to Σ_1 then we say that Σ_1 and Σ_2 are *locally isometric.*

Proposition 8. *Suppose $\phi_1 : U \to \Sigma_1$ and $\phi_2 : U \to \Sigma_2$ are coordinate charts on Σ_1 and Σ_2 so that*

$$g^1_{11} = g^2_{11}, \quad g^1_{12} = g^2_{12}, \quad g^1_{22} = g^2_{22}.$$

Then

$$\phi_2 \circ \phi_1^{-1}$$

is a local isometry.

Conversely, if $\phi_1 : U \to \Sigma_1$ is a coordinate chart and

$$\psi : \phi_1(U) \to \Sigma_2$$

is a local isometry then ϕ_1 and $\psi \circ \phi_1$ have the same coefficients g_{ij} for their first fundamental forms.

Proof:

Take $p \in \Sigma_1$, and $X \in T_p\Sigma_1$. Let $\alpha : (-\epsilon, \epsilon) \to U \subset \mathbb{R}^2$ be a curve with $\phi_1 \circ \alpha(0) = p$, $(\phi_1 \circ \alpha)'(0) = X$. Then

$$X = (u^1)'(0)(\phi_1)_{u^1} + (u^2)'(0)(\phi_1)_{u^2}$$

whilst $d(\phi_2 \circ \phi_1^{-1})_p(X)$ is the velocity vector to the curve

$$\phi_2 \circ \phi_1^{-1} \circ \phi_1 \circ \alpha = \phi_2 \circ \alpha.$$

So

$$d(\phi_2 \circ \phi_1^{-1})_p(X) = (u^1)'(0)(\phi_2)_{u^1} + (u^2)'(0)(\phi_2)_{u^2}.$$

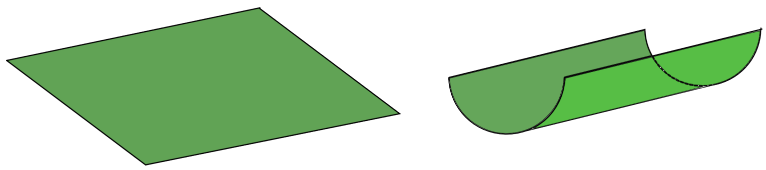
$$\begin{aligned} I_p(X) &= (u^1)'(0)g^1_{11} + 2(u^1)'(0)(u^2)'(0)g^1_{12} + (u^2)'(0)g^1_{22} \\ I_{(\phi_2\circ\phi_1^{-1})p}(d(\phi_2 \circ \phi_1^{-1}))_p(X) &= (u^1)'(0)g^2_{11} + 2(u^1)'(0)(u^2)'(0)g^2_{12} + (u^2)'(0)g^2_{22} \end{aligned}$$

So $\phi_2 \circ \phi_1^{-1}$ is a local isometry if and only if

$$g^1_{11} = g^2_{11},\ g^1_{12} = g^2_{12},\ g^1_{22} = g^2_{22}.$$

□

Example 9. The cylinder and the plane are locally isometric.



Example 9 (continued). Several weeks ago (Lecture 11) we parametrised the plane in $\mathbb{R}^3$ through the point p_0 containing the orthonormal cectors v, w by

$$\phi(u^1, u^2) = p_0 + u^1 v + u^2 w,$$

and computed

$$g_{11} \equiv 1, \quad g_{12} \equiv 0, \quad g_{22} \equiv 1.$$

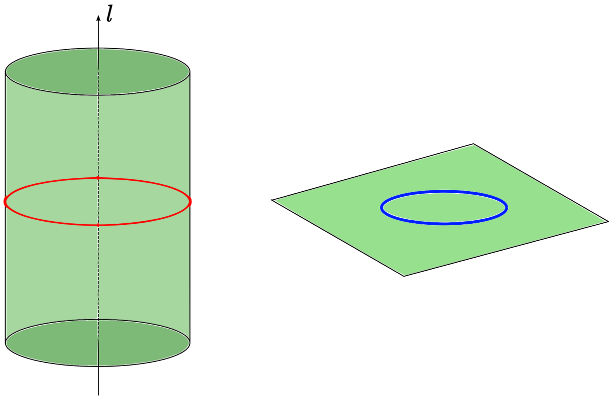
We also computed that for $v, w \in \mathbb{R}^3$ orthonormal, l the line with direction vector $v \times w$ through the point p_0, and Σ the cylinder of radius 1 about the line l, the local parametrisation

$$\begin{aligned} \phi : \quad (0, 2\pi) \times \mathbb{R} &\to \Sigma \\ (u^1, u^2) &\mapsto p_0 + \cos(u^1)v + \sin(u^1)w + u^2(v \times w) \end{aligned}$$

gives the same g_{ij}.

Example 9 (continued). However, the cylinder and the plane are *not* isometric, because they are not homeomorphic.

A cylinder contains closed curves which cannot be contracted within it to a point; any closed curve within the plane can be.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