MATH3968 Lecture 25

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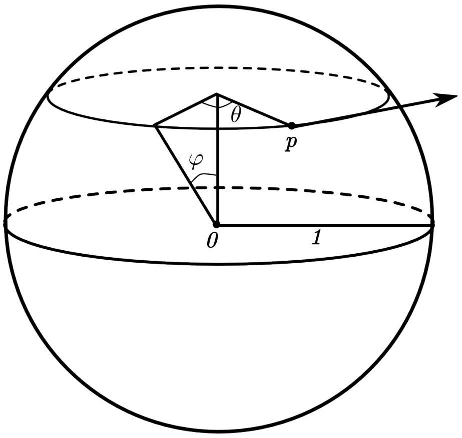
Parallel Transport along a Circle of Constant Latitude on $\mathbb{S}^2$



Example 1.

Example 1 (continued). Spherical coordinates give the parameterisation

$$\phi(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).$$

Then

$$g_{11} = \sin^2\varphi, \quad g_{12} = 0, \quad g_{22} = 1.$$

Example 1 (continued). Now we need to calculate the Christoffel symbols. We will derive the general formula, and write everything using Einstein's summation convention. Recall that

$$(\mathbf{E}_i)_j = \Gamma^k_{ij}\mathbf{E}_k + h_{ij}\boldsymbol{N}.$$

So

$$\begin{aligned}\frac{\partial}{\partial u^k} g_{ij} &= \langle (\mathbf{E}_i)_{u^k}, \mathbf{E}_j\rangle + \langle \mathbf{E}_i, (\mathbf{E}_j)_{u^k}\rangle \\ &= \Gamma^m_{ik} g_{mj} + \Gamma^m_{jk} g_{mi}\end{aligned}$$

Example 1 (continued). Using commutativity of partial derivatives ($\Gamma^m_{ij} = \Gamma^m_{ji}$),

$$\begin{aligned}\frac{\partial}{\partial u^k} g_{ij} + \frac{\partial}{\partial u^i} g_{jk} &- \frac{\partial}{\partial u^j} g_{ki} \\ &= \Gamma^m_{ik} g_{mj} + \Gamma^m_{jk} g_{mi} + \Gamma^m_{ji} g_{mk} + \Gamma^m_{ki} g_{mj} - \Gamma^m_{kj} g_{mi} - \Gamma^m_{ij} g_{mk} \\ &= 2\Gamma^m_{ik} g_{mj}\end{aligned}$$

Example 1 (continued).

$$\left(\frac{\partial}{\partial u^k} g_{ij} + \frac{\partial}{\partial u^i} g_{jk} - \frac{\partial}{\partial u^j} g_{ki}\right) g^{jl} = 2\Gamma^m_{ik} g_{mj} g^{jl} = 2\Gamma^m_{ik}\delta^l_m = 2\Gamma^l_{ik}$$

so

$$\Gamma^l_{ik} = \frac{1}{2} g^{jl}\left(\frac{\partial}{\partial u^k} g_{ij} + \frac{\partial}{\partial u^i} g_{jk} - \frac{\partial}{\partial u^j} g_{ki}\right)$$

Example 1 (continued). This all agrees with what we had before, namely

$$\begin{aligned}
\Gamma^1_{11}g_{11} + \Gamma^2_{11}g_{12} &= \frac{1}{2}(g_{11})_{u^1} \\
\Gamma^1_{11}g_{12} + \Gamma^2_{11}g_{22} &= (g_{12})_{u^1} - \frac{1}{2}(g_{11})_{u^2} \\
\Gamma^1_{12}g_{11} + \Gamma^2_{12}g_{12} &= \frac{1}{2}(g_{11})_{u^2} \\
\Gamma^1_{12}g_{12} + \Gamma^2_{12}g_{22} &= \frac{1}{2}(g_{22})_{u^1} \\
\Gamma^1_{22}g_{11} + \Gamma^2_{22}g_{12} &= (g_{12})_{u^2} - \frac{1}{2}(g_{22})_{u^1} \\
\Gamma^1_{22}g_{12} + \Gamma^2_{22}g_{22} &= \frac{1}{2}(g_{22})_{u^2},
\end{aligned}$$

but writing the equations in terms of Γ^k_{ij} rather than individual indices makes it less messy to write down the solution.

Example 1 (continued). For the sphere, the only nonzero Christoffel symbols are

$$\Gamma^1_{12} = \cot\varphi, \quad \Gamma^2_{11} = -\sin\varphi\cos\varphi.$$

We computed earlier that the covariant derivative of the vector field $w = a^i\mathbf{E}_i$ along the vector $v = b^j\mathbf{E}_j$ is

$$\nabla_v(w) = \left(\frac{da^k}{dt} + a^i b^j \Gamma^k_{ij}\right)\mathbf{E}_k.$$

Reminder:

$$\begin{aligned}
(dw)(v) &= \frac{da^i}{dt}\mathbf{E}_i + a^i\frac{d\mathbf{E}_i}{dt} \\
&= \frac{da^i}{dt}\mathbf{E}_i + a^i b^j(\Gamma^k_{ij}\mathbf{E}_k + h_{ij}N),
\end{aligned}$$

so the projection of this to the tangent space is given by the above formula.

Example 1 (continued). We parameterise a circle of constant latitude by $\theta = t, \varphi = \varphi_0$, so

$$b^1 = 1, \quad b^2 = 0.$$

Thus our equations for parallel transport are

$$\begin{aligned}
\frac{da^1}{d\theta} + a^2\cot\varphi_0 &= 0 \\
\frac{da^2}{d\theta} - a^1\sin\varphi_0\cos\varphi_0 &= 0
\end{aligned}$$

or, substituting $\tilde{a}^1 = a^1\sin\varphi_0$,

$$\begin{aligned}
\frac{d\tilde{a}^1}{d\theta} &= \cos\varphi_0 a^2 \\
\frac{da^2}{d\theta} &= \cos\varphi_0 a^1
\end{aligned}$$

Example 1 (continued). These have solution

$$\begin{pmatrix} \tilde{a}^1(\theta) \\ a^2(\theta) \end{pmatrix} = \begin{pmatrix} \cos((\cos\varphi_0)\theta) & \sin((\cos\varphi_0)\theta) \\ -\sin((\cos\varphi_0)\theta) & \cos((\cos\varphi_0)\theta) \end{pmatrix} \begin{pmatrix} \tilde{a}^1(0) \\ a^2(0) \end{pmatrix}$$

Example 1 (continued). Both methods then give us that if you parallel translate the vector w through an angle θ along the lines of constant latitude $\varphi = \varphi_0$, it rotates through an angle of

$$(\cos\varphi_0)\theta.$$

http://torus.math.uiuc.edu/jms/java/dragsphere/ has a nice Java applet where you can view this parallel translation.

Definition 2. A nonconstant parameterised curve $\alpha : I \to \Sigma$ is a *parameterised geodesic* if

$$\nabla_{\alpha'(t)}\alpha'(t) = 0 \quad \text{for all } t \in I,$$

i.e. if its tangent vector field is parallel along the curve.

Proposition 3. *If w_1, w_2 are parallel vector fields along a curve α, then $\langle w_1, w_2\rangle$ is constant along α. In particular, parallel translation preserves angle and length.*

Question 4. Prove this.

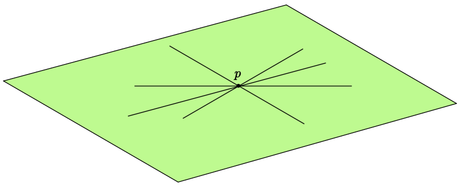
Answer 5.

$$\begin{aligned} \frac{d}{dt}\langle w_1, w_2\rangle &= \left\langle \frac{dw_1}{dt}, w_2 \right\rangle + \left\langle w_1, \frac{dw_2}{dt} \right\rangle \\ &= \left\langle \frac{Dw_1}{dt}, w_2 \right\rangle + \left\langle w_1, \frac{Dw_2}{dt} \right\rangle \\ &\quad \text{since } w_1, w_2 \text{ are tangent vector fields} \\ &= 0 \quad \text{by the parallel assumption.} \end{aligned}$$

Thus a parameterised geodesic has tangent vectors of constant length, and so is parameterised proportionally to arc-length.

Definition 6. A regular unparameterised curve α in Σ is called a *geodesic* if its parameterisation by arc-length is a parameterised geodesic.

In the plane, the geodesics are exactly the straight lines; the curves of zero curvature.



Since the covariant derivative depends only on the 1st fundamental form and its derivatives, geodesics are sent to geodesics under isometries: