MATH3968 Lecture 34

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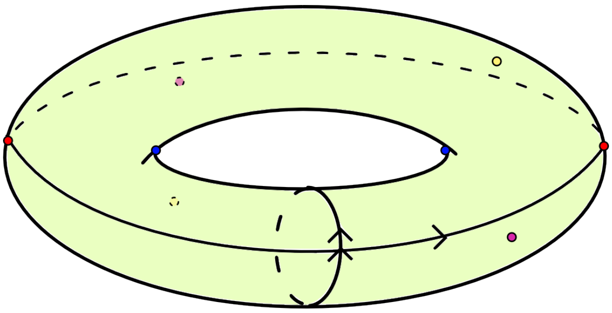
Example 1 (Klein bottle). Let T be a torus of revolution

$$z^2 = R^2 - (\sqrt{x^2 + y^2} - a)^2$$

in $\mathbb{R}^3$, and define an equivalence relation on T by

$$(x, y, z) \sim (-x, -y, -z) = A(x, y, z).$$

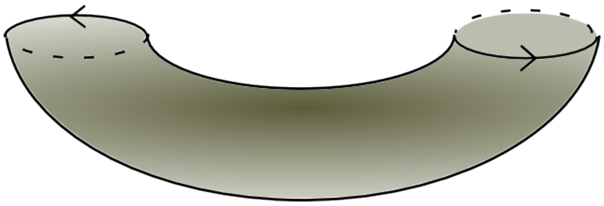


Example 1 (continued).

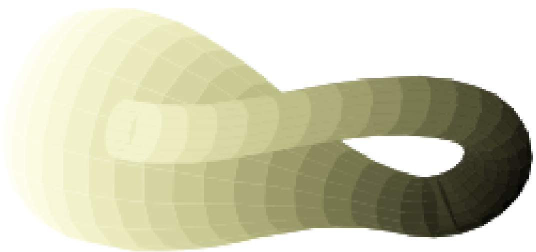
Example 1 (continued). Write $\pi : T \to \frac{T}{\sim}$, and cover T with coordinate charts $(U_\alpha, \varphi_\alpha)$ so that

$$\varphi_\alpha(U_\alpha) \bigcap A(\varphi_\alpha(U_\alpha)) = \emptyset$$

Then $(U_\alpha, \pi \circ \varphi_\alpha)$ defines $\pi : T \to \frac{T}{\sim}$ as an abstract surface, called the *Klein bottle*.

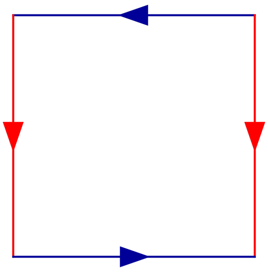


Example 1 (continued).



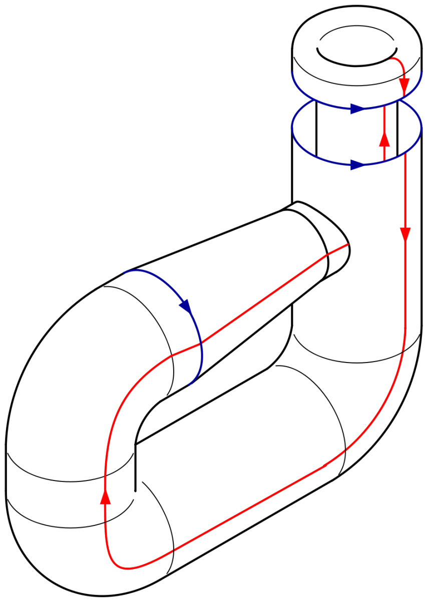
Example 1 (continued).

Example 1 (continued). Another way to construct a Klein bottle is



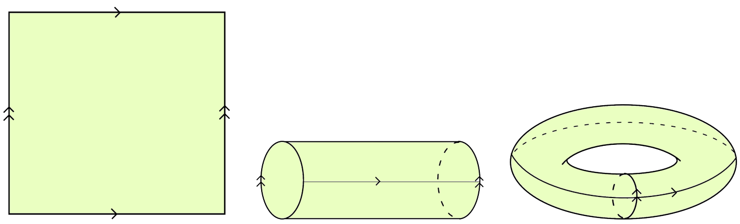
Let $R = [0,1] \times [0,1]$ and define an equivalence relation $\sim$ on R by

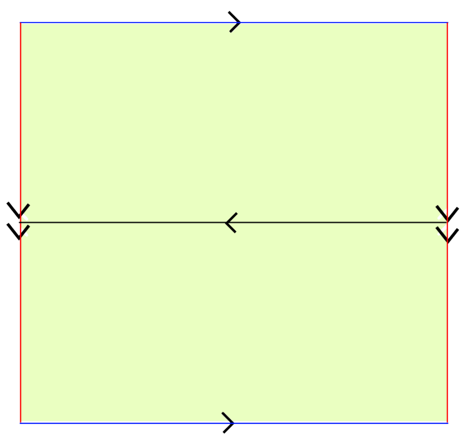
- $(0, y) \sim (1, y)$
- $(x, 0) \sim (1 - x, 1)$



Example 1 (continued).

Example 1 (continued). Note that





Example 1 (continued).

Recall that we defined a tangent vector to a regular surface Σ in $\mathbb{R}^3$ at $p \in \Sigma$ to be the velocity vector $\alpha^{'}(0)$ to a smooth parameterised curve $\alpha : (-\epsilon, \epsilon) \to \Sigma$ with $\alpha(0) = p$.

We then defined the tangent space $T_p\Sigma$ to be the set of all tangent vectors.

For an abstract surface/manifold we no longer have an ambient space in which to compute a velocity vector.

However, we can still talk about curves α:

Definition 2. A smooth parameterised curve in a (smooth) manifold M is a smooth map $\alpha : (a, b) \to M$.

What we mean by saying that α is smooth is that each $\alpha \circ \varphi^{-1}$ is smooth whenever the range of α and the domain of φ^{-1} have non-empty intersection. This is covered in the definitions above, since $(0, 1)$ is a smooth one-dimensional manifold.

We can't define tangent vectors to be curves, because in the old situation of surfaces in $\mathbb{R}^3$, two different curves could have the same velocity vector.

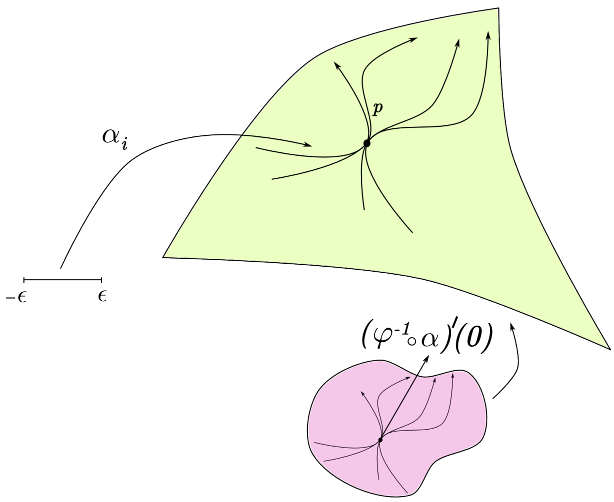
What we can do is

Definition 3 (Tangent Vectors As Equivalence Classes of Curves)**.** A *tangent vector* v to the smooth manifold M at $p \in M$ is an equivalence class $[\alpha]$ of curves $\alpha : (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$.

Fix a coordinate chart $\varphi : U \to M$ around p. If $\alpha_1, \alpha_2 : (-\epsilon, \epsilon) \to M$ satisfy $\alpha_1(0) = \alpha_2(0) = p$ we define

$$\alpha_1 \sim \alpha_2 \iff (\varphi^{-1} \circ \alpha_1)'(0) = (\varphi^{-1} \circ \alpha_2)'(0).$$

This is independent of the choice of φ.



Alternatively, we can think of a tangent vector as an operator that acts on functions to give a directional derivative.

For a regular surface Σ in $\mathbb{R}^3$, a tangent vector $\alpha'(0) \in T_p\Sigma$ acts on functions f that are smooth at p by

$$\alpha^{'}(0)(f) = \frac{d(f \circ \alpha)}{dt}\Big|_{t=0}.$$

This suggests defining a tangent vector as an operator on functions; it will have to satisfy the product rule, and of course be linear.

Definition 4 (Derivations at p)**.** Let M be a smooth manifold, $p \in M$, and denote by $C^\infty(M)$ the real vector space of smooth functions on M. A *derivation at p M* is a map

$$D : C^\infty(M) \to \mathbb{R}$$

which

1. is linear, ie $D(f+g) = D(f) + D(g)$, $D(cf) = cD(f)$, $f, g \in C^\infty(M), c \in \mathbb{R}$.
2. satisfies the product rule at p:

$$D(fg) = f(p)D(g) + g(p)D(f).$$

Notice that for surfaces in $\mathbb{R}^3$, the directional derivative

$$\alpha^{'}(0)(f) = \frac{d(f \circ \alpha)}{dt}\Big|_{t=0}.$$

determined by the tangent vector $\alpha'(0)$ is indeed a derivation.