MATH3968 Lecture 35

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Alternatively, we can think of a tangent vector as an operator that acts on functions (to give a "directional derivative").

Definition 1 (Derivations at p)**.** Let M be a smooth manifold, $p \in M$, and denote by $C^\infty(M)$ the real vector space of smooth functions on M. A *derivation at p M* is a map

$$D : C^\infty(M) \to \mathbb{R}$$

which

1. is linear, ie $D(f+g) = D(f) + D(g)$, $D(cf) = cD(f)$, $f, g \in C^\infty(M), c \in \mathbb{R}$.
2. satisfies the product rule at p:

$$D(fg) = f(p)D(g) + g(p)D(f).$$

Notice that for surfaces in $\mathbb{R}^3$, the directional derivative

$$\alpha^{'}(0)(f) = \frac{d(f \circ \alpha)}{dt}\Big|_{t=0}.$$

determined by the tangent vector $\alpha'(0)$ is indeed a derivation.

Notation: Previously, we have used the notation

$$E_i = \frac{\partial \varphi}{\partial u^i}.$$

When thinking of tangent vectors as operators, it is convenient to drop the explicit reference to the coordinate chart, and write

$$\frac{\partial}{\partial u^i} : f \mapsto \frac{\partial f(p)}{\partial u^i}.$$

Similarly, it is convenient to write a function f on M as $f(u^1, \ldots, u^n)$ rather than $f(\varphi(u^1, \ldots, u^n))$.

Notice that in all these notations, the point p is implicit.

Proposition 2. *The set of all derivations at $p \in M^n$ forms a n-dimensional vector space, with basis*

$$\frac{\partial}{\partial u^i} : f \mapsto \frac{\partial f(p)}{\partial u^i},$$

where

$$\begin{array}{rcl} \varphi : U \subset \mathbb{R}^n & \to & M \\ (u^1, \ldots, u^n) & \mapsto & \varphi(u^1, \ldots, u^n) \end{array}$$

is a local parametrisation of the smooth manifold M near p, with U a convex open neighbourhood of $0 \in \mathbb{R}^n$ and $\varphi(0) = p$.

Proof: It is easy to check that a linear combination of derivations is still a derivation.

Proof (con'd):

The $\frac{\partial}{\partial u^i}$ are linearly independent; if $a^i \frac{\partial}{\partial u^i} = 0$ then

$$0 = a^i \frac{\partial}{\partial u^i}(u_j) = a^j \text{ for each } j = 1, \dots, n.$$

It remains to show that any derivation at p can be written as a linear combination of the $\frac{\partial}{\partial u^i}$, with coefficients as described above.

Lemma 3. *Let $f : \varphi(U) \to \mathbb{R}$ be a smooth function with $f(p) = 0$.*

Then there are smooth functions $a_i : \varphi(U) \to \mathbb{R}$, such that

1. $f(\varphi(u^1, \dots, u^n)) = a_i u^i$ *using Einstein summation convention*
2. $a_i(p) = \dfrac{\partial(f \circ \varphi)}{\partial u^i}(0)$.

The second condition follows from the first, by differentiating with respect to u^i and evaluating at $u = (0, \dots, 0)$.we achieve simpler notation by omitting the φ and writing

$$a_i(p) = \frac{\partial f}{\partial u^i}(p).$$

Proof:

Fix $u \in U$. Since $(0, \dots, 0) \in U$ and U is convex (contains the line between any two points in it), $tu \in U$ for $0 \leq t \leq 0$.

Define

$$h_u(t) = f(\varphi(tu)), \quad 0 \leq t \leq 0.$$

Then

$$f(\varphi(u)) = f(\varphi(u)) - f(\varphi(0)) = \int_0^1 h'_u(t)\, dt = \int_0^1 u^i \frac{\partial(f \circ \varphi)}{\partial u^i}\, dt$$

so we may take

$$a_i = \int_0^1 \frac{\partial(f \circ \varphi)}{\partial u^i}\, dt.$$

□

We now show that any derivation D at p can be written as a linear combination of the $\frac{\partial}{\partial u^i}$.

First notice that

$$D(1) = D(1 \cdot 1) = 1D(1) + 1D(1)$$

so $D(1) = 0$.Then also $D(c) = cD(1) = 0$ for any constant function c.

Using the above lemma,

$$\begin{aligned} D(f) - D(f(0)) &= D(f) \\ &= D(a_i u^i) \\ &= D(a_i)u^i(p) + a_i(p)D(u^i) \\ &= \frac{\partial f}{\partial u^i}(p). \end{aligned}$$

□

Proposition 4 (Tangent Vectors As Derivations)**.** *There is a one to one correspondence between equivalence classes of curves through $p \in M$ (as defined above) and derivations at p. Thus we can alternatively define a tangent vector at p to be a derivation at p.*

Proof: Given an equivalence class of curves $[\alpha]$, choose a representative $\alpha : (-\epsilon, \epsilon) \to M$ and define $D_{[\alpha]}$ by

$$D_{[\alpha]}(f) := \frac{d(f \circ \alpha)}{dt}\Big|_{t=0}.$$

Check (tutorial exercise) that this is well-defined (independent of the choice of representative α), 1-1, and that D is a derivation at p, i.e.

$$D(f+g) = D(f) + D(g), \quad D(cf) = cD(f), \quad D(fg) = f(p)D(g) + g(p)D(f).$$

Given a derivation D at p, we would like to define a curve α so that D corresponds to differentiating along α at p.

Recall that for regular surfaces in $\mathbb{R}^3$, if $\alpha(t) = \varphi(u^1(t), u^2(t))$, with $\alpha(0) = p$,

$$\begin{aligned} \alpha'(0) &= {u^1}'(0)\boldsymbol{E_1} + {u^2}'(0)\boldsymbol{E_2} \\ &= D_{[\alpha]}(u^1)\boldsymbol{E_1} + D_{[\alpha]}(u^2)\boldsymbol{E_2} \end{aligned}$$

The simplest curve with this derivative is given by using a straight line:

$$\beta(t) = \varphi(D_{[\alpha]}(u^1)t, D_{[\alpha]}(u^2)t)$$

where we have assumed that $\varphi(0,0) = p$.

Motivated by this, define

$$\alpha_D(t) = \varphi(D(u^1)t, \ldots, D(u^n)t).$$

Question 5. Show that $[\alpha_{D_\alpha}] = [\alpha]$, $D_{[\alpha_D]} = D$.

To prove $D_{[\alpha_D]} = D$, by our lemma above, it suffices to evaluate on the functions $u^i : M \to \mathbb{R}$:

$$\begin{aligned} D_{[\alpha_D]}(u^i) &= \frac{u^i \circ \alpha_D}{dt}\Big|_{t=0} \\ &= \frac{u^i \circ \varphi(D(u^1)t, \ldots, D(u^n)t)}{dt}\Big|_{t=0} \\ &= D(u^i) \end{aligned}$$

We have shown that every derivation acts as a directional derivative

$$D_{[\alpha]}(f) := \frac{d(f \circ \alpha)}{dt}\Big|_{t=0}.$$

for some α. □

Corollary 7. *Let M be a smooth manifold, $p \in M$, $\varphi : U \subset \mathbb{R}^n \to M$ a coordinate chart around p, and $u^1, \ldots, u^n$ the standard coordinates in $\mathbb{R}^n$.*

Define the tangent space T_pM to M to be the set of all tangent vectors to M at p. Then T_pM is an n-dimensional vector space, with basis vectors

$$\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n}.$$

Example 8. The simplest n-dimensional manifold is $\mathbb{R}^n$, with the identity map as a single coordinate chart.

Then a basis for $T_p\mathbb{R}^n$ (any p) is

$$\frac{\partial}{\partial u^i} : f \mapsto \frac{\partial f(p)}{\partial u^i},$$

or as equivalence classes of curves,

$$[(t, 0, \ldots, 0)], [(0, t, 0, \ldots, 0)], \ldots, [(0, \ldots, 0, t].$$

Example 9 (Real projective space, $\mathbb{RP}^n$). Recall that $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Equivalently,

$$\mathbb{RP}^n = \frac{\mathbb{R}^{n+1} \setminus \{0\}}{\sim},$$

where

$$(X_0, X_1, \ldots, X_n) \sim \lambda(X_0, X_1, \ldots, X_n),\ \lambda \in \mathbb{R} \setminus \{0\}.$$

Write the equivalence class of $(X_0, X_1, \ldots, X_n)$ as $[X_0 : X_1 : \ldots : X_n]$ Let

$$U_i = \{[X_0 : X_1 : \ldots : X_n] : X_i \neq 0\},\ i = 0, 1, \ldots, n$$

and note that $X_i = 0$ is well-defined even though $X_i =$ any other number is not.

Example 9 (continued). Every equivalence class in U_0 has a unique representative with $X_0 = 1$.

Define

$$\begin{array}{rccc} \varphi_0 : & \mathbb{R}^n & \to & U_0 \\ & (u^1, u^2, \ldots, u^n) & \mapsto & [1 : u^1 : u^2 : \ldots : u^n] \end{array}$$

Similarly, define

$$\begin{array}{rccc} \varphi_i : & \mathbb{R}^n & \to & U_i \\ & (u^1, u^2, \ldots, u^n) & \mapsto & [u^1 : \ldots : 1 : \ldots u^n] \end{array} \text{ (1 in } i\text{th position)}$$

Example 9 (continued). We can always use a coordinate chart about $p \in M$ to give the basis

$$\frac{\partial}{\partial u^i} : f \mapsto \frac{\partial f(p)}{\partial u^i},$$

of T_pM.

For $p \in U_0 \subset \mathbb{RP}^n$, we have a coordinate chart

$$\begin{array}{rccc} \varphi_0 : & \mathbb{R}^n & \to & U_0 \\ & (u^1, u^2, \ldots, u^n) & \mapsto & [1 : u^1 : u^2 : \ldots : u^n] \end{array}$$

so as equivalence classes of curves, we can take the curves with representatives

$$[1 : t : 0 : \ldots : 0], [1 : 0 : t : 0 : \ldots : 0], \ldots, [1 : 0 : \ldots : 0 : t].$$

Active Learning

If

$$F : M_1 \to M_2$$

is smooth at $p \in M_1$, we can define the differential

$$dF_p : T_pM_1 \to T_{F(p)}M_2$$

by either

$$dF_p([\alpha]) = [F \circ \alpha]$$

or

$$dF_p(v) : f_2 \mapsto v(f_2 \circ F),$$

depending on which definition of tangent vectors we use.

Question 10. Check that these definitions are equivalent.