The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial Week 2

MATH3968: Differential Geometry — Semester 2, 2009

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"Lecture notes" refers to *Lecture Notes for Differential Geometry, MATH3968* by Nigel O'Brien.
"do Carmo" refers to *Differential Geometry of Curves and Surfaces*, by Manfredo do Carmo.
Solutions to exercises in the class notes are posted separately; below are solutions to the remaining exercises.

Required Problems

1. do Carmo §1.5 Q12 and Lecture Notes Exercise Set 2, Q5

Solution:

a. By the chain rule, $\frac{ds}{dt}\frac{dt}{ds} = 1$. Since $s = s(t)$ is arc length along α, $\frac{ds}{dt} = |\alpha'|$. Thus

$$\frac{dt}{ds} = \frac{1}{|\alpha'|}.$$

We need to differentiate this again with respect to s, which will involve differentiating $|\alpha'|$. Recall that $|\alpha'| = \langle \alpha', \alpha' \rangle^{1/2}$. In addition, for any $u, v : \mathbb{R} \to \mathbb{R}^3$,

$$\begin{aligned}\frac{d}{dt}\langle u, v\rangle &= \frac{d}{dt}(u_1v_1 + u_2v_2 + u_3v_3) \\ &= u_1'v_1 + u_1v_1' + u_2'v_2 + u_2v_2' + u_3'v_3 + u_3v_3' \\ &= \langle u', v\rangle + \langle u, v'\rangle.\end{aligned}$$

Thus we have

$$\begin{aligned}\frac{d}{dt}|\alpha'| &= \frac{d}{dt}\langle \alpha', \alpha'\rangle^{1/2} \\ &= \frac{\langle \alpha'', \alpha'\rangle + \langle \alpha', \alpha''\rangle}{2\langle \alpha', \alpha'\rangle^{1/2}} \\ &= \frac{\langle \alpha', \alpha''\rangle}{|\alpha'|}.\end{aligned}$$

What we actually want is:

$$\begin{aligned}
\frac{d^2t}{ds^2} &= \frac{d}{ds}\frac{1}{|\alpha'|} \\
&= \frac{dt}{ds}\frac{d}{dt}\frac{1}{|\alpha'|} \\
&= \frac{dt}{ds}\left(-\frac{1}{|\alpha'|^2}\right)\frac{d}{dt}|\alpha'| \\
&= \frac{1}{|\alpha'|}\left(-\frac{1}{|\alpha'|^2}\right)\frac{\langle\alpha',\alpha''\rangle}{|\alpha'|} \\
&= -\frac{\langle\alpha',\alpha''\rangle}{|\alpha'|^4}.
\end{aligned}$$

b. Recall the definition of curvature:

$$k(s) = \left|\frac{d^2\alpha(s)}{ds^2}\right|.$$

One solution is given in example 2.5 of the class notes, but the following solution enables one to see how the answer could have been arrived at without being given the formula first. We have

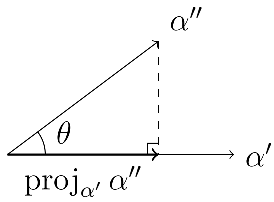
$$\begin{aligned}
\frac{d^2\alpha}{ds^2} &= \frac{d}{ds}\left(\frac{dt}{ds}\frac{d\alpha}{dt}\right) \\
&= \frac{d^2t}{ds^2}\frac{d\alpha}{dt} + \frac{dt}{ds}\frac{d^2\alpha}{dt^2}\frac{dt}{ds} \\
&= -\frac{\langle\alpha',\alpha''\rangle}{|\alpha'|^4}\alpha' + \frac{\alpha''}{|\alpha'|^2},
\end{aligned}$$

Therefore,

$$k = \left|\frac{d^2\alpha}{ds^2}\right| = \frac{1}{|\alpha'|^2}\left|\alpha'' - \frac{\langle\alpha',\alpha''\rangle}{|\alpha'|^2}\alpha'\right|.$$

If the angle between the vectors α' and α'' is θ, then recall that the projection of α'' onto α' is

$$\operatorname{proj}_{\alpha'}\alpha'' = (|\alpha''|\cos\theta)\frac{1}{|\alpha'|}\alpha' = \frac{\langle\alpha',\alpha''\rangle}{|\alpha'|^2}\alpha'.$$



Therefore, the vector $\alpha'' - \frac{\langle\alpha',\alpha''\rangle}{|\alpha'|^2}\alpha'$ has length $|\alpha''|\sin\theta = |\alpha' \times \alpha''|/|\alpha'|$. It follows that

$$k = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}.$$

c. Recall the vector triple product, a scalar quantity depending on three vectors (u, v and w) which gives the (oriented) volume of the parallelepiped spanned by those vectors. The base has area $|u \times v|$ and the perpendicular height is $|w| \cos\theta$ (where θ is the angle between w and $u \times v$), so the volume is $\langle u \times v, w \rangle$, which is called the vector triple product of u, v and w. This value is cyclically symmetric in u, v and w:

$$\langle u \times v, w \rangle = \langle v \times w, u \rangle = \langle w \times u, v \rangle.$$

See the solutions to question 5 of exercise sheet 2 in the class notes.

d. The absolute value of the signed curvature was found in part b:

$$|k(t)| = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{|(x'y'' - x''y')\mathbf{k}|}{\sqrt{(x')^2 + (y')^2}^3} = \frac{|x'y'' - x''y'|}{((x')^2 + (y')^2)^{3/2}}.$$

Let θ be the angle traversed anticlockwise going from the vector α' to α''. We know that $\alpha' \times \alpha'' = (x'y'' - x''y')\mathbf{k}$ points in the positive z-direction if and only if $\sin\theta > 0$, i.e. θ is in the range $(0, \pi)$—in other words, if and only if (α', α'') is a *positively* oriented basis of $\mathbb{R}^2$. Thus $x'y'' - x''y' > 0$ if and only if (α', α'') is a positively oriented basis of $\mathbb{R}^2$, which is true exactly when the signed curvature $k(t)$ is positive. Therefore

$$k(t) = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}}.$$

2. Lecture Notes, Exercise Set 2, Q1bc, Q2, Q3

3. do Carmo §5.7 p404 Q1bc

Solution:

a. -3.

b. -4.

c. 1.

d. 0.

4. Let A be an invertible $n \times n$ matrix with real entries.

(a) Show that the following are equivalent:

(i) $AA^t = I$, $A^tA = I$;

(ii) for all $v, w \in \mathbb{R}^n$, $\langle Av, Aw \rangle = \langle v, w \rangle$;

(iii) the columns of A form an orthonormal basis for $\mathbb{R}^n$;

(iv) the rows of A form an orthonormal basis for $\mathbb{R}^n$.

Such matrices are called orthogonal, and we denote the set of all such matrices by $O(n)$.

(b) Show that $O(n)$ is a subgroup of the group of invertible $n \times n$ matrices under matrix multiplication, i.e. it contains the identity, the product of any two orthogonal matrices is orthogonal, and the inverse of an orthogonal matrix is orthogonal.

(c) Show that an orthogonal matrix has determinant either 1 or -1.

An orthogonal matrix with determinant 1 is called special orthogonal; the set of such matrices is denoted $SO(n)$ and is also a group.

Solution:

(a) **i $\implies$ ii.** For any $v, w \in \mathbb{R}^n$, considering them as column vectors, we have

$$\langle Av, Aw\rangle = (Av)^t Aw = v^t A^t Aw = v^t I w = v^t w = \langle v, w\rangle.$$

ii $\implies$ iii. Denote by e_i the ith standard basis vector. As the standard basis is orthonormal, $\langle e_i, e_j\rangle = \delta_{ij}$ for all i, j. The ith column of A is Ae_i, and by hypothesis $\langle Ae_i, Ae_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}$ for all i, j, so the columns of A are orthonormal. Since A has n columns, they form an orthonormal basis of $\mathbb{R}^n$.

iii $\implies$ i. The (i, j)th entry of A^tA is the inner product of the ith and jth columns of A. Since the columns of A are orthonormal, this value is δ_{ij}. Hence $A^tA = I$ and consequently $AA^t = I$ also.

i $\iff$ iv. We have the following chain of equivalences: $AA^t = I \iff A^t(A^t)^t = I \iff$ the columns of A^t are an orthonormal basis of $\mathbb{R}^n \iff$ the rows of A are an orthonormal basis of $\mathbb{R}^n$. In the second equivalence we used the fact that i $\iff$ iii, established above.

(b) Since $II^t = I$, $I \in O(n)$, so $O(n)$ contains the identity. If $A, B \in O(n)$ then $AA^t = I$ and $BB^t = I$, so $AB(AB)^t = ABB^tA^t = AIA^t = AA^t = I$, so $AB \in O(n)$. In other words, the product of any two orthogonal matrices is orthogonal. For inverses, note that $(A^t)^{-1} = (A^{-1})^t$ (since $A^t(A^{-1})^t = (A^{-1}A)^t = I^t = I$). Thus if $AA^t = I$ then also $A^{-1}(A^{-1})^t = A^{-1}(A^t)^{-1} = (A^tA)^{-1} = I^{-1} = I$, so $A^{-1} \in O(n)$. Thus the inverse of an orthogonal matrix is orthogonal. This establishes that $O(n)$ is a subgroup of the group of $n \times n$ invertible matrices.

(c) Suppose A is orthogonal. Then $AA^t = I$. Taking determinants of both sides, we get $\det(A)\det(A^t) = \det(I) = 1$. But $\det(A^t) = \det(A)$ by definition of the determinant. Thus $\det(A)^2 = 1$, so $\det(A) = 1$ or -1.

Recommended Problems

5. Lecture Notes, Exercise Set 1, Q6

6. do Carmo §1.5 p22 Q1cde

Solution: It was computed in lectures that s equals arc length, and the tangent, normal, binormal and curvature are given by

$$\begin{aligned}
\mathbf{t}(s) &= \frac{1}{c}\left(-a\sin\frac{s}{c}, a\cos\frac{s}{c}, b\right),\\
\mathbf{n}(s) &= \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right),\\
\mathbf{b}(s) &= \mathbf{t}(s)\times\mathbf{n}(s) = \frac{b}{c}\sin\frac{s}{c}\mathbf{i} - \frac{b}{c}\cos\frac{s}{c}\mathbf{j} + \frac{a}{c}\mathbf{k},\\
k &= \frac{a}{c^2}.
\end{aligned}$$

c. The osculating plane is the plane through $\alpha(s)$ perpendicular to $\mathbf{b}(s)$ as given above.

d. Note that these lines are in the direction $\mathbf{n}(s)$ and furthermore that $\langle \mathbf{n}(s), \mathbf{k} \rangle = 0$ by the above. Therefore the lines are always perpendicular to the x-axis.

e. The tangent lines are in the direction $\mathbf{t}(s)$. Since $\mathbf{t}(s)$ is a unit vector, the quantity $\mathbf{t}(s) \cdot \mathbf{k}$ is equal to the sine of the angle which the tangent line to α at $\alpha(s)$ makes with the x-axis. But $\mathbf{t}(s) \cdot \mathbf{k} = b$ as above, which is a constant, so the tangent lines to α make a constant angle with the x-axis.

7. do Carmo §5.7 p405 Q5

Solution:

a. Suppose that p_0q were tangent to C at q. By definition of a convex curve, all points of C lie in a closed half-plane bounded by p_0q. The interior of C must therefore lie in the interior of that half-plane, which does not contain p_0 (as p_0 lies on the boundary of the half-plane). This contradicts the fact that p_0 is in the interior of C.

b. Let $\alpha : [0, l] \to \mathbb{R}^2$ be a regular parameterisation of C. Let $\phi : [0, l] \to \mathbb{R}$ and $\theta : [0, l] \to \mathbb{R}$ be continuous maps such that the vectors $q(s) - p_0$ and $\alpha'(s)$ point in the directions at angles $\theta(s)$ and $\phi(s)$ respectively (measured anticlockwise from $\mathbf{i}$) for all $s \in [0, l]$. Note that $q(s) - p_0$ and $\alpha'(s)$ are collinear for no $s \in [0, l]$, by part a, so we know that $\theta(s) - \phi(s)$ is never a multiple of π.

By adding or subtracting multiples of 2π if necessary, we may insist that $\theta(0) - \phi(0)$ lie in the range $(-\pi, \pi)$. Since $\theta(s) - \phi(s)$ is never a multiple of π, it follows that $\theta(s) - \phi(s) \in (-\pi, \pi)$ for all $s \in [0, l]$. In particular, $\theta(l) - \phi(l) \in (-\pi, \pi)$.

However, $\theta(0)$ and $\theta(l)$ represent the same physical direction, so they must differ by a multiple of 2π, and the same is true of $\phi(0)$ and $\phi(l)$. Therefore $\theta(l) - \phi(l)$ differs from $\theta(0) - \phi(0)$ by a multiple of 2π. We conclude that $\theta(l) - \phi(l) = \theta(0) - \phi(0)$. Hence $\theta(l) - \theta(0) = \phi(l) - \phi(0)$. Dividing by 2π, we see that the winding number of C relative to p_0 equals the rotation index of C.

c. This question is false as stated—one must assume additionally that C is simple. (Without this assumption, we could simply trace out the same closed curve repeatedly.) Assuming that C is simple, let α, p_0 and θ be as in part b. If ever $\theta(s) = \theta(s') + 2k\pi$, for some $s \neq s' \in [0, l)$ and $k \in \mathbb{Z}$, then we must have $\alpha(s) \neq \alpha(s')$ (as C is simple); hence one of $\alpha(s)$ and $\alpha(s')$ will lie outside the half-plane bounded by the tangent at the other, contradicting that C is convex. Therefore $\theta([0, l))$ lies within an interval of length at most 2π. It follows that the winding number of C about p_0, which is $(\theta(l) - \theta(0))/2\pi$, must be 0, 1 or -1.

Furthermore θ is monotonic, because it is injective on the interval $[0, l)$ and continuous. Therefore the winding number cannot be 0; it must be 1 or -1. As noted in part b, the rotation index of C is equal to this winding number.