

### Solutions to Tutorial Week 3

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MATH3968: Differential Geometry

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Lecturer: Emma Carberry

“Lecture Notes” refers to *Lecture Notes for Differential Geometry, MATH3968* by Nigel O’Brien. “do Carmo” refers to *Differential Geometry of Curves and Surfaces*, by Manfredo do Carmo.

Solutions to exercises in the class notes are posted separately; below are solutions to the remaining exercises.

#### Required Problems

1. Lecture Notes, Exercise Set 3, Q1,2c,5.
2. Recall the following definition:

**Definition:** Let  $U \subset \mathbb{R}^n$  be open and  $f : U \rightarrow \mathbb{R}^m$  be smooth. The *differential of  $f$  at  $p \in U$*  is a linear map

$$df_p : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

defined as follows: take  $w \in \mathbb{R}^n$ , and let  $\alpha : (-\varepsilon, \varepsilon) \rightarrow U$  be a smooth curve so that  $\alpha(0) = p$ , and  $\alpha'(0) = w$ . Then

$$df_p(w) = (f \circ \alpha)'(0).$$

Using this definition, prove the following proposition:

**Proposition:**  $df_p$  is a well-defined linear map, and with respect to the standard bases is given by the matrix

$$df_p := \begin{pmatrix} f_1^1(p) & f_2^1(p) & \cdots & f_n^1(p) \\ f_1^2(p) & f_2^2(p) & \cdots & f_n^2(p) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^m(p) & f_2^m(p) & \cdots & f_n^m(p) \end{pmatrix},$$

where  $f_j^i = \frac{\partial f^i}{\partial x_j}$ .

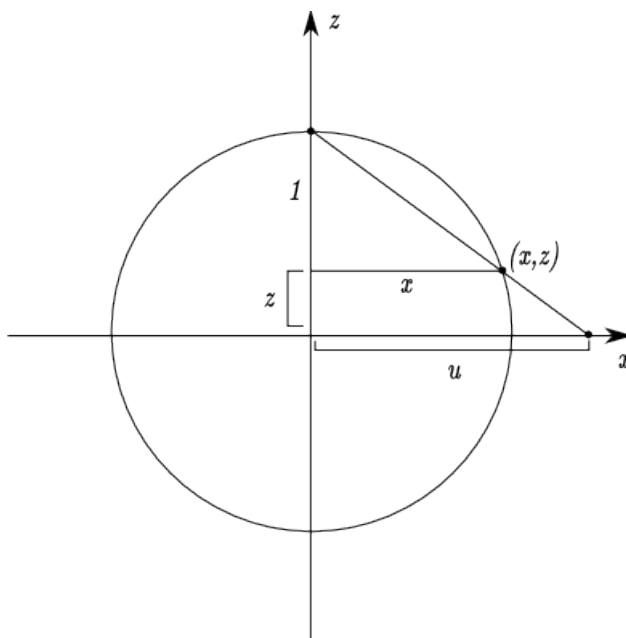
**Solution:** Refer to do Carmo p127-8, where this proposition is stated and proved.

3. Show that the inverse of the stereographic projection from the North Pole of the sphere to the plane through its equator is given by

$$\begin{aligned}x &= \frac{2u}{u^2 + v^2 + 1} \\y &= \frac{2v}{u^2 + v^2 + 1} \\z &= \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\end{aligned}$$

**Solution:**

Consider the following diagram, showing a cross-section of the sphere in the  $xz$ -plane:



Using a similar triangles argument, one can show that

$$\frac{1}{u} = \frac{1 - z}{x}, \tag{1}$$

and a similar argument in the  $yz$ -plane shows that

$$\frac{1}{v} = \frac{1 - z}{y}. \tag{2}$$

Square (1) and (2), add, and use the defining relation for the sphere, namely

$x^2 + y^2 + z^2 = 1$ , to get

$$\begin{aligned}u^2 + v^2 &= \frac{x^2 + y^2}{(1 - z)^2} \\&= \frac{1 - z^2}{(1 - z)^2} \\&= \frac{(1 - z)(1 + z)}{(1 - z)^2} \\&= \frac{1 + z}{1 - z}\end{aligned}$$

Multiply both sides by  $1 - z$  and simplify:

$$\begin{aligned}(1 - z)(u^2 + v^2) &= (1 + z) \\z(1 + u^2 + v^2) &= 1 - u^2 - v^2 \\z &= \frac{1 - u^2 - v^2}{1 + u^2 + v^2}\end{aligned}$$

Substituting this expression for  $z$  into (1) and (2) will then give the required forms for  $x$  and  $y$  in terms of  $u$  and  $v$ , as required.

### Recommended Problems

4. Prove that a linear map  $A : \mathbb{R}^n \rightarrow \mathbb{R}^{n+m}$  is injective if and only if it has rank  $n$ .

**Solution:** Let  $e_1, \dots, e_n$  be a basis for  $\mathbb{R}^n$  and assume first that  $A : \mathbb{R}^n \rightarrow \mathbb{R}^{n+m}$  is injective. This is equivalent to saying that the rank of  $A$ ,  $\text{rk}(A) = \dim(A(\mathbb{R}^n))$ , the dimension of the image of  $A$ . We need them to show that  $\dim(A(\mathbb{R}^n)) = n$ .

We first claim that  $A(e_1), \dots, A(e_n)$  are linearly independent. To show this, suppose that  $0 = \sum_{i=1}^n \lambda_i A(e_i)$  for some  $\lambda_i \in \mathbb{R}^n$ . Then

$$\begin{aligned}0 &= \sum_{i=1}^n \lambda_i A(e_i) \\&= \sum_{i=1}^n A(\lambda_i e_i) \\&= A\left(\sum_{i=1}^n \lambda_i e_i\right),\end{aligned}$$

where the last two equalities follow by the linearity of  $A$ . Now since  $A$  is injective we must have  $\sum \lambda_i e_i = 0$ , and so  $\lambda_i = 0$  for all  $i$  since  $e_i$  is a basis, and so  $A(e_i)$  are linearly independent, meaning  $\text{span}(A(e_i))$  is  $n$ -dimensional.

To show that  $\text{span}(A(e_i)) = A(\mathbb{R}^n)$ , consider

$$\text{span}(A(e_i)) = \left\{ \sum_{i=1}^n \lambda_i A(e_i) \mid \lambda_i \in \mathbb{R}^n \right\} = \left\{ A\left(\sum_{i=1}^n \lambda_i e_i\right) \mid \lambda_i \in \mathbb{R}^n \right\} = A(\mathbb{R}^n),$$

as required.

Conversely, assume that  $A$  has rank  $n$ , i.e. that  $A(e_i)$  are linearly independent. Suppose that  $A(x) = A(y)$  for some  $x, y \in \mathbb{R}^n$ , so that  $A(x - y) = 0$  by linearity of  $A$ . Now,  $x - y = \sum_{i=1}^n \lambda_i e_i$  for some  $\lambda_i \in \mathbb{R}^n$  since  $e_i$  is a basis for  $\mathbb{R}^n$ . So

$$\begin{aligned} 0 &= A(x - y) \\ &= A\left(\sum_{i=1}^n \lambda_i e_i\right) \\ &= \sum_{i=1}^n \lambda_i A(e_i). \end{aligned}$$

But since  $A(e_i)$  are linearly independent, we must have  $\lambda_i = 0$  for all  $i$ , so  $x - y = 0$ , giving  $x = y$ . So  $A$  is injective.

5. Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

can be solved for  $x, y, u$  in terms of  $z$ ; for  $x, z, u$  in terms of  $y$ , for  $y, z, u$  in terms of  $x$ ; but not for  $x, y, z$  in terms of  $u$ .

**Solution:** Since the equations are linear in  $x, y, z$ , we can solve for  $x, y, z$  in terms of  $u$  if and only if the coefficient matrix

$$A := \begin{pmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{pmatrix}$$

is invertible. A quick calculation shows  $\det A = 0$ , and hence we cannot.

We claim however that we can solve for  $x, y, u$  in terms of  $z$ . We use the implicit function theorem to show this.

Define a function

$$f(x, y, z, u) = \begin{pmatrix} 3x + y - z + u^2 \\ x - y + 2z + u \\ 2x + 2y - 3z + 2u \end{pmatrix}.$$

Now,

$$df = \begin{pmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{pmatrix}.$$

The implicit function theorem tells us that if

$$\det \begin{pmatrix} 3 & 1 & 2u \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \neq 0,$$

then we can solve for  $x, y, u$  in terms of  $z$ . Another quick calculation gives that the determinant of the above matrix is  $-12 + 8u$ , which is nonzero if  $u \neq \frac{3}{2}$ .

We show that when  $u = \frac{3}{2}$ , the equations

$$A\mathbf{x} = \mathbf{b}$$

have no solution, where

$$\mathbf{x} := \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} := \begin{pmatrix} -u^2 \\ -u \\ -2u \end{pmatrix}.$$

This is an exercise in Gauss-Jordan elimination (see first year) which, after row and column operations, results in the augmented matrix

$$\left( \begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{7}{4} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

A similar method applies to the other two cases.

**6.** Lecture Notes, Exercise Set 3, Q3,4.