

## Solutions to Tutorial Week 5

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MATH3968: Differential Geometry

Semester 2, 2009

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### Required Problems

1. Let  $a, b, c$  be nonzero real numbers. Show that the three equations

$$\begin{aligned}x^2 + y^2 + z^2 &= ax, \\x^2 + y^2 + z^2 &= by, \quad \text{and} \\x^2 + y^2 + z^2 &= cz\end{aligned}$$

each define regular surfaces, and that these intersect orthogonally.

c.f. do Carmo §2.4 Q12.

**Solution:** Assume that  $a, b, c$  are all nonzero. Let  $f(x, y, z) = x^2 + y^2 + z^2 - ax$ ,  $g(x, y, z) = x^2 + y^2 + z^2 - bx$  and  $h(x, y, z) = x^2 + y^2 + z^2 - cx$ . (These define spheres in  $\mathbb{R}^3$ .)

Then  $df(x, y, z)$  is the column vector  $(2x - a, 2y, 2z)^T$ , and is 0 only at  $(\frac{a}{2}, 0, 0)$ . Since  $f(\frac{a}{2}, 0, 0) = -\frac{a^2}{4} \neq 0$  the first surface is regular. Moreover the normal direction is given by the gradient vector  $\nabla(f)(x, y, z) = (2x - a)\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ . (Note the similarity with  $df$ ! This is always the case for surfaces defined by an equation  $f = 0$ .) Similarly for the others.

The first two surfaces intersect at points where  $ax = by$ . At such a point  $\nabla(f) \bullet \nabla(g) = 4x^2 - 2ax + 4y^2 - 2by + 4z^2 = 0$ .

Similarly for the other pairs. (All three surfaces intersect simultaneously at  $(0, 0, 0)$  and at the point  $(\frac{t}{a}, \frac{t}{b}, \frac{t}{c})$ , where  $t = (a^{-2} + b^{-2} + c^{-2})^{-1}$ .)

2. Lecture Notes, Exercise Set 4, Q5
3. Lecture Notes, Exercise Set 5, Q5
4. Lecture Notes, Exercise Set 6, Q1

### Recommended Problems

5. Lecture Notes, Exercise Set 4, Q4
6. do Carmo §2.4 p88 Q20, Q22

**Solution:** See do Carmo *Hints and Answers* pp 482-3