The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial Week 7

MATH3968: Differential Geometry Semester 2, 2009

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"Lecture Notes" refers to *Lecture Notes for Differential Geometry, MATH3968* by Nigel O'Brien. "do Carmo" refers to *Differential Geometry of Curves and Surfaces*, by Manfredo do Carmo.

Solutions to exercises in the class notes are posted separately; below are solutions to the remaining exercises.

Required Problems

1. Show that the sum of the normal curvatures for any pair of orthogonal directions at a given point $p \in S$ is $2H(p)$.

Solution: Let v_1 and v_2 be unit vectors in the principal directions. If v is any other unit vector in $T_P(S)$ and $\langle v, v_1\rangle_P = \cos\theta$ then $v = \cos\theta v_1 + \sin\theta v_2$.

Hence $k_n(v) = II_P(v) = k_1\cos^2\theta + k_2\sin^2\theta$.

If $w \perp v$ is another unit vector in $T_P(S)$ then $w = \pm\sin\theta v_1 \pm \cos\theta v_2$.

Hence $k_n(v) + k_n(w) = k_1(\cos^2\theta + \sin^2\theta) + k_2(\sin^2\theta + \cos^2\theta) = k_1 + k_2 = 2H$.

2. do Carmo §3.2 p152 Q13

Solution:

Consider a point p on the curve in the surface. The normal vector N will also be normal to the osculating plane spanned by $\{\mathbf{t}, \mathbf{n}\}$ of the curve at this point. Now N will point in the same direction as $\mathbf{n} \times \mathbf{t}$ (at least up to the sign of the orientation, which we can choose, so no generality is lost), and so by the Frenet equations we see

$$N' = (\mathbf{n} \times \mathbf{t})' = \mathbf{b}' = \tau\mathbf{n}.$$

Using inner product notation, and taking lengths,

$$\begin{aligned}
\tau^2 &= |N'|^2 = \langle (N\circ\alpha)'(s), (N\circ\alpha)'(s)\rangle \\
&= \langle dN_p(\alpha'(s)), dN_p(\alpha'(s))\rangle \\
&= \big\langle (dN_p)^2(\alpha'(s)), \alpha'(s)\big\rangle \quad \text{since } dN_p \text{ is self-adjoint} \\
&= \big\langle (dN_p)^2(\cos\theta\mathbf{e_1} + \sin\theta\mathbf{e_2}, \cos\theta\mathbf{e_1} + \sin\theta\mathbf{e_2}\big\rangle \\
&= k_1^2\cos^2\theta + k_2^2\sin^2\theta. \qquad (*)
\end{aligned}$$

Now since the curve is asymptotic, we have

$$k_n = k_1\cos^2\theta + k_2\sin^2\theta = 0. \qquad (1)$$

This implies $k_1 \cos^2 \theta = -k_2 \sin^2 \theta$, and so $k_1^2 \cos^2 \theta = -k_1 k_2 \sin^2 \theta$, $k_2^2 \sin^2 \theta = -k_1 k_2 \cos^2 \theta$ and substituting these relations into (*) we get

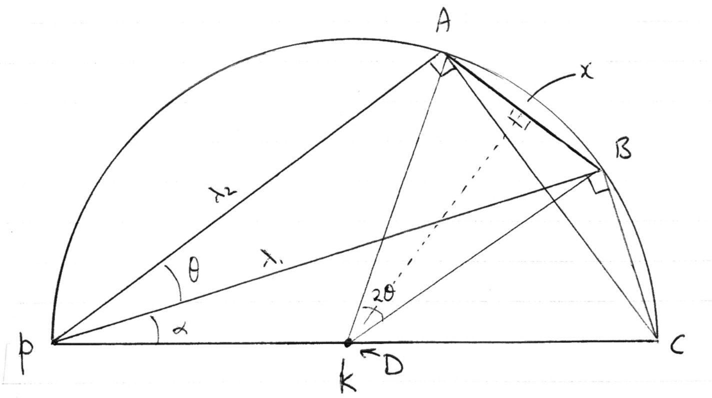
$$\tau^2 = k_1^2 \cos^2 \theta + k_2^2 \sin^2 \theta = -k_1 k_2 \sin^2 \theta - k_1 k_2 \cos^2 \theta = -k_1 k_2 = -K,$$

i.e. $\tau = \sqrt{-K}$, as required.

3. do Carmo §3.2 p151 Q14

Solution: See do Carmo *Hints and Solutions* pp 483-4 for a solution involving the vector triple product.

For a more crafty solution which explains how the expression gets its cosine-rule-like appearance, consider the following diagram.



$\mathbf{n}$, N_1 and N_2 must all lie in the same plane by the geometry of the situation. Consider the vector $k\mathbf{n}$ with length k pointing in the direction of $\mathbf{n}$, and the vector $\lambda_1 N_1$ (resp. $\lambda_2 N_2$) with length λ_1 (resp. λ_2) pointing in the direction of N_1 (resp. N_2), where λ_1 and λ_2 are the normal curvatures at p. The angle between $\lambda_1 N_1$ and $\lambda_2 N_2$ is θ; call the angle between $\lambda_1 N_1$ and $k\mathbf{n}$ α. We can form the right-angled triangles because the diameter of a circle subtends a right angle at the circumference (high-school circle geometry).

Consider the line labelled x in the diagram. Using the cosine rule, we find

$$x^2 = \lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2 \cos \theta. \tag{2}$$

We now find the length of x a different way. Form the triangle labelled ABD in the diagram. We know (again from high school circle geometry) that the angle θ at the circumference of the circle will be half the angle $\angle ADB$ at the centre. So we label this angle 2θ, and bisect it creating the two right triangles shown in the diagram.

Since the line AD is a radius of the circle, it has length $\frac{k}{2}$, and so again by elementary trigonometry we get $\frac{x}{2} = \frac{k}{2} \sin \theta$, giving

$$x = k \sin \theta.$$

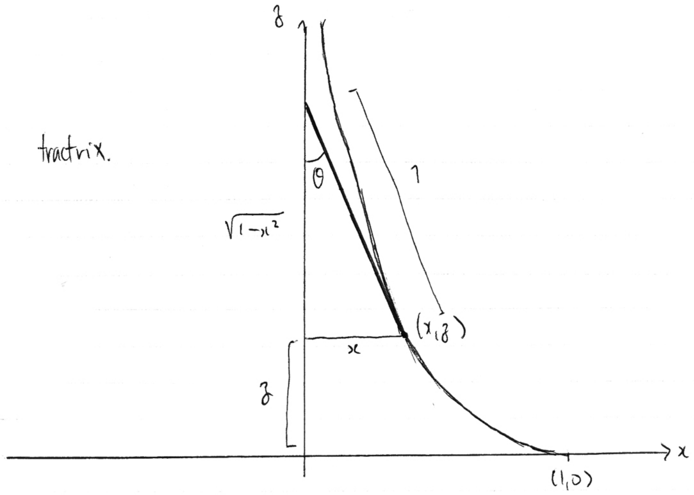
Substituting this into (2) we get the required result, namely

$$k^2 \sin^2 \theta = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \cos \theta.$$

Thanks to Andy for this enlightening solution.

4. do Carmo §3.3 p168 Q6

Solution:



(a) From the diagram, we see

$$\frac{dz}{dx} = \frac{\sqrt{1-x^2}}{x}.$$

We obtain a parametrization for the tractrix by setting $x = \sin \theta$ (as θ is labelled above) and integrating:

$$z = \int \frac{\cos \theta}{\sin \theta} \cos \theta d\theta = \int \frac{\cos^2 \theta}{\sin \theta} d\theta.$$

To solve this integral we use the method of t-substiution. This is simply an exercise in recalling double-angle formulas. So let $t = \tan\frac{\theta}{2}$. Then

$$\begin{aligned}
\cos\theta &= \cos\left(2\frac{\theta}{2}\right) = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \\
&= 2\cos^2\frac{\theta}{2} - 1 = \frac{2}{\sec^2\frac{\theta}{2}} - 1 \\
&= \frac{2}{1-\tan^2\frac{\theta}{2}} - 1 = \frac{1-t^2}{1+t^2}. \\
\sin\theta &= \sin\left(2\frac{\theta}{2}\right) = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot\frac{\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}} \\
&= \frac{2\sin^2\frac{\theta}{2}}{\tan\frac{\theta}{2}} \\
&= \frac{2\tan^2\frac{\theta}{2}}{\tan\frac{\theta}{2}\left(1+\tan^2\frac{\theta}{2}\right)} = \frac{2t}{1+t^2} \\
dt &= \frac{1}{2}\sec^2\frac{\theta}{2}d\theta = \frac{1}{2}(1+t^2)d\theta \\
d\theta &= \frac{2}{1+t^2}dt.
\end{aligned}$$

So

$$\begin{aligned}
z = \int\frac{\cos^2\theta}{\sin\theta}d\theta &= \int\frac{1-t^2}{1+t^2}\cdot\frac{1+t^2}{2t}\cdot\frac{2}{1+t^2}dt \\
&= \int\frac{1-2t^2+t^4}{t+2t^3+t^5}dt.
\end{aligned}$$

An online integrator then gives

$$z = \log t + \frac{2}{t^2+1} + C.$$

Now

$$\frac{2}{1+t^2} = \frac{1-t^2}{1+t^2} + \frac{1+t^2}{1+t^2},$$

and so

$$z = \log t + \frac{1-t^2}{1+t^2} + C + 1 = \log t + \cos\theta + C'.$$

We can eliminate this constant by noticing that $\theta = \frac{\pi}{2}$ when $z = 0$, and so $C' = 0$. Hence our parametrized curve is

$$\begin{aligned}
z &= \log\left(\tan\frac{\theta}{2}\right) + \cos\theta, \\
x &= \sin\theta,
\end{aligned}$$

where $0 < \theta \leq \frac{\pi}{2}$.

(b) We parametrize our surface of revolution by

$$\mathbf{x} = \begin{cases} x = \sin\theta\cos\phi \\ y = \sin\theta\sin\phi \\ z = \log\left(\tan\frac{\theta}{2}\right) + \cos\theta \end{cases}$$

where $0 < \theta \leq \frac{\pi}{2}$ and $0 < \phi \leq 2\pi$. We only really need to check the regularity condition. Consider the differential

$$d\mathbf{x} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \cos\phi\cos\theta & -\sin\phi\sin\theta \\ \sin\phi\cos\theta & \cos\phi\sin\theta \\ \frac{1-\sin^2\theta}{\sin\theta} & 0 \end{pmatrix}.$$

(For calculation of $\frac{dz}{d\theta}$ see below. For $\theta \neq \frac{\pi}{2}$, the determinant of

$$\begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \cos\phi\cos\theta & -\sin\theta\sin\phi \\ \sin\phi\cos\theta & \sin\theta\cos\phi \end{pmatrix}$$

is nonzero and hence $d\mathbf{x}$ has rank 2.

When $\theta = \frac{\pi}{2}$,

$$\begin{aligned} \frac{dz}{d\theta} &= \frac{\sec^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}} - \sin\theta \\ &= \frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}} \cdot \frac{1}{\cos^2\frac{\theta}{2}} - \sin\theta = \frac{1}{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}} - \sin\theta \\ &= \frac{1}{\sin\theta} - \sin\theta = \frac{1-\sin^2\theta}{\sin\theta} \end{aligned}$$

is 0 and so

$$d\mathbf{x}_{\left(\frac{\pi}{2},\phi\right)} = \begin{pmatrix} 0 & -\sin\phi \\ 0 & \cos\phi \\ 0 & 0 \end{pmatrix}$$

does not have maximal rank at $\theta = \frac{\pi}{2}$, and so the surface is not regular at $\left(\frac{\pi}{2},\phi\right)$, i.e. at the points where $z = 0$ on the "equator" of the pseudosphere; but it is regular elsewhere.

(c) The simplest method appears to be calculating

$$\frac{z'(x'z'' - x''z')}{x(x'^2 + z'^2)},$$

where z and x refer to the parametric equations found in Part (a) and where a prime denotes the derivative with respect to θ. This formula was derived in lectures for a surface of revolution. This is quite a laborious calculation.

5. Lecture Notes, Exercise Set 7, Q3

Recommended Problems

6. Lecture Notes, Exercise Set 7, Q2

Note: Sectional curvature is what we have been referring to in lectures as normal curvature. An elliptic point is one where the Gauss curvature $K(p) > 0$, or equivalently, the principal curvatures have the same sign.

7. do Carmo §3.3 p168 Q16

Solution: See do Carmo *Hints and Solutions* p485.