THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial Week 8

MATH3968: Differential Geometry — Semester 2, 2009

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"Lecture Notes" refers to *Lecture Notes for Differential Geometry, MATH3968* by Nigel O'Brien. "do Carmo" refers to *Differential Geometry of Curves and Surfaces*, by Manfredo do Carmo.

Solutions to exercises in the class notes are posted separately; below are solutions to the remaining exercises.

Required Problems

1. do Carmo §3.5 p209 Q14

Solution:

(a) Use the following parametrisation for the helicoid

$$f = (a\sinh v \sin u,\ -a\sinh v\cos u,\ au),$$

and the following for the catenoid

$$g = (a\cosh v\cos u,\ a\cosh v\sin u,\ av),$$

and it is easy to show

$$\frac{\partial f}{\partial u} = \frac{\partial g}{\partial v}, \quad \frac{\partial f}{\partial v} = -\frac{\partial g}{\partial u},$$

as required.

(b) We can assume $\mathbf{x}$ and $\mathbf{y}$ are covered with conformal coordinate charts, since they are minimal. We thus first need to show that the coordinate chart on $\mathbf{z}$ induced by these is conformal as well, i.e. we need to show $g^z_{11} = g^z_{22}$ and $g^z_{21} = 0$.

Now $g^x_{11} = \langle x_u, x_u\rangle = \langle x_v, x_v\rangle = g^x_{22}$, and similarly $g^y_{11} = \langle y_u, y_u\rangle = \langle y_v, y_v\rangle = g^y_{22}$, since our coordinates are conformal.

We also know $g^x_{12} = \langle x_u, x_v\rangle = 0 = \langle y_u, y_v\rangle = g^y_{12}$, also because our coordinates are conformal.

Finally, since $\mathbf{x}$ and $\mathbf{y}$ are conjugate minimal, we know

$$x_u = y_v, \quad x_v = -y_u.$$

Using all this, it is simple to show

$$\begin{aligned}\langle z_u, z_u\rangle = g^z_{11} &= \cos^2 t\,\langle x_u, x_u\rangle + \sin^2 t\,\langle y_u, y_u\rangle = g^z_{22}\\ \langle z_u, z_v\rangle = g^z_{12} &= 0.\end{aligned}$$

So indeed our induced coordinates are conformal. Note we have omitted the boldface above and used x and $\mathbf{x}$ interchangeably, and similarly for y and $\mathbf{y}$. Finally we need to show $\Delta z \equiv (0,0,0)$, but this follows immediately by the minimality of $\mathbf{x}$ and $\mathbf{y}$, and the linearity of the Laplacian operator, since

$$\Delta \mathbf{z} = \cos t \Delta \mathbf{x} + \sin t \Delta \mathbf{y} = \cos t(0,0,0) + \sin t(0,0,0) = (0,0,0),$$

and so $\mathbf{z}$ is indeed a minimal surface as required.

(c) Notice above we can replace

$$g_{11}^{z} = \cos^2 t \left\langle x_u, x_u \right\rangle + \sin^2 t \left\langle y_u, y_u \right\rangle$$

with

$$g_{11}^{z} = \cos^2 t \left\langle x_u, x_u \right\rangle + \sin^2 t \left\langle x_v, x_v \right\rangle = \cos^2 t \left\langle x_u, x_u \right\rangle + \sin^2 t \left\langle x_u, x_u \right\rangle = \left\langle x_u, x_u \right\rangle$$

as required. The expression for g_{22}^{z} follows similarly. We have already shown in (b) that $g_{12}^{z} = 0$.

2. do Carmo §4.2 p227 Q3

Solution: See do Carmo *Hints and Solutions* p487.

3. do Carmo §4.2 p227 Q7

Solution: We prove (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (b) $\Rightarrow$ (a). We also note that the question should have included $\dim V = \dim W$, otherwise the question doesn't make sense.

(a)$\Rightarrow$(c) Suppose $\langle F(v_1), F(v_2)\rangle = \langle v_1, v_2\rangle$ for all $v_1, v_2 \in V$. Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis in V. Then

$$\langle v_i, v_j\rangle = \langle F(v_i), F(v_j)\rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

proving $\{F(v_1), \ldots, F(v_n)\}$ is an orthonormal set.

To prove it is a basis, we need only to show that $\{F(v_1), \ldots, F(v_n)\}$ are linearly independent, since any linearly independent set of n elements will be a basis in an n-dimensional vector space (see MATH2961). So assume $\sum_{i=1}^{n} \lambda_i F(v_i) = 0$. Then

$$\begin{aligned} \left(\sum_{i=1}^{n} \lambda_i F(v_i)\right)^2 = 0 \quad &\Rightarrow \quad \sum_{i=1}^{n} \lambda_i^2 \left|F(v_i)\right|^2 = 0 \\ &\Rightarrow \quad \sum_{i=1}^{n} \lambda_i^2 \left|v_i\right|^2 = 0 \\ &\Rightarrow \quad \left(\sum_{i=1}^{n} \lambda_i v_i\right)^2 = 0 \\ &\Rightarrow \quad \lambda_i = 0 \end{aligned}$$

for all i since the v_i's are linearly independent.

(c)⇒(d) Every vector space has a basis. We can always make this basis orthonormal by Gram-Schmidt orthogonalisation. By (c) if $\{v_1, \ldots, v_n\}$ is such a basis then $\{F(v_1), \ldots, F(v_n)\}$ will also be an orthonormal basis for W.

(d)⇒(b) Let $\{v_i\}$ be an orthonormal basis of V such that $\{F_i\}$ is an orthnormal basis of W. Take $v \in V$. Then $v = \sum \lambda_i v_i$ for some λ_i. Then

$$|v|^2 = \langle v, v \rangle = \left\langle \sum \lambda_i v_i, \sum \lambda_i v_i \right\rangle = \sum_{i,j} \lambda_i \lambda_j \langle v_i, v_j \rangle = \sum_i \lambda_i^2$$

since $\{v_i\}$ are orthonormal. Similarly

$$|F(v)|^2 = \langle F(v), F(v) \rangle = \left\langle \sum \lambda_i F(v_i), \sum \lambda_i F(v_i) \right\rangle = \sum_i \lambda_i^2$$

by linearity of F and since $\{F(v_i)\}$ are orthonormal.

(b)⇒(a) If $|F(v)| = |v|$ for all $v \in V$ then $\langle F(v), F(v) \rangle = \langle v, v \rangle$ for all $v \in V$.

4. do Carmo §4.2 p227 Q11

Solution: Remember the definition of distance-preserving from Q8, and use Q3.

For a solution see do Carmo, *Hints and Solutions* p488.

Recommended Problems

5. do Carmo §3.5 p209 Q12

Solution: Refer to do Carmo §3.3, Q16 (p172).

A compact surface by this question must have an elliptic point; and so the Gauss curvature is positive somewhere on the surface, hence the mean curvature cannot everywhere vanish (the condition for a minimal surface. Hence there are no compact minimal surfaces.