The University of Sydney
School of Mathematics and Statistics

Tutorial Week 3

MATH3968: Differential Geometry — Semester 2, 2009

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"Lecture Notes" refers to *Lecture Notes for Differential Geometry, MATH3968* by Nigel O'Brien. "do Carmo" refers to *Differential Geometry of Curves and Surfaces*, by Manfredo do Carmo.

Required Problems

1. Lecture Notes, Exercise Set 3, Q1,2c,5.

2. Recall the following definition:

Definition: Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be smooth. The *differential of f at $p \in U$* is a linear map

$$df_p : \mathbb{R}^n \to \mathbb{R}^m$$

defined as follows: take $w \in \mathbb{R}^n$, and let $\alpha : (-\varepsilon, \varepsilon) \to U$ be a smooth curve so that $\alpha(0) = p$, and $\alpha'(0) = w$. Then

$$df_p(w) = (f \circ \alpha)'(0).$$

Using this definition, prove the following proposition:

Proposition: *df_p is a well-defined linear map, and with respect to the standard bases is given by the matrix*

$$df_p := \begin{pmatrix} f_1^1(p) & f_2^1(p) & \cdots & f_n^1(p) \\ f_1^2(p) & f_2^2(p) & \cdots & f_n^2(p) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^m(p) & f_2^m(p) & \cdots & f_n^m(p) \end{pmatrix},$$

where $f_j^i = \frac{\partial f^i}{\partial x_j}$.

3. Show that the inverse of the stereographic projection from the North Pole of the sphere to the plan through its equator is given by

$$\begin{aligned} x &= \frac{2u}{u^2+v^2+1} \\ y &= \frac{2u}{u^2+v^2+1} \\ z &= \frac{u^2+v^2-1}{u^2+v^2+1} \end{aligned}$$

Recommended Problems

4. Prove that a linear map $A : \mathbb{R}^n \to \mathbb{R}^{n+m}$ is injective if and only if it has rank n.

5. Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y, for y, z, u in terms of x; but not for x, y, z in terms of u.

6. Lecture Notes, Exercise Set 3, Q3,4.